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ELEMENTARY PARTICLE STRUCTURE

CHAPTER I

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The purpose of this discussion and analysis is to explore one possible theoretical approach to the general problem of correlating the elementary particles. This problem, so phrased, is of immediate interest because of the increasing number of known particles, but apart from the modern phrasing ("correlating the elementary particles") this same philosophical problem has been with us for several thousand years.

Man's search for the "elements," for knowledge of the ultimate structure of matter, has led him, stage by stage, to the concept of an elementary particle, the irreducible structural unit of matter. Yet it has not been many years since the days when the nucleus was considered inviolate, and a few years earlier it was the atom that was thought to be indivisible. Now today, as the list of elementary particles continues to lengthen, more and more physicists share the suspicion that the elementary particles may not be truly ultimate and structureless after all.

If man's ideas about the structure of matter were collected and arranged in rough historical sequence, then the later part of the list might look like this:

(v) Molecules

(w) Atoms

(x) Nuclei (and electrons)

(y) Elementary particles: n, p, e, ν, γ, μ, π, τ, κ, θ, χ, Λ, ...

It can be seen that the historical order is also a rough order of decreasing size. Furthermore, this historical sequence of ideas is, in addition, a structural sequence of conceptual levels, in the sense that the members of each level are constructed from building blocks found on the next lower level.

If the elementary particles are not to be considered ultimate and structureless, that means that there exists at least one conceptual level still lower on the list. For example such a level might include as members a limited selection from the elementary particles, the rest of the elementary particles then being composite structures built from this limited number. Such a

point of view would divide the elementary particles into two categories, one group being considered more fundamental than the other. Examples of this point of view are Louis de Broglie's proposal[1] that the photon may be a combination of two neutrinos, and the suggestion by Fermi and Yang[2] that the pi-meson may be built from a nucleon and an anti-nucleon. According to a second point of view[3], the next lower level might consist entirely of entities not found (or not yet found) among the elementary particles. There could also be a third point of view, in which the next lower conceptual level is supposed to contain new entities in addition to certain selected elementary particles.

However, as soon as the existence of a next lower level has been established, we will have to face the question whether there is a still lower level beneath, and so on. Will this sequence of conceptual levels ever terminate, or will it continue indefinitely? Can we always expect to find further internal structure as we refine our theories or our measuring apparatus, or is there a last level, whose members have no inner structure?

A tentative answer to this question can be phrased in the following way. If any level in the sequence contains as many as two distinguishable members, then it is meaningful to ask about their inner structure, to ask whether there is a further level below But if all of the elementary particles (along with any other phenomena susceptible to direct measurement) can be constructed from a single entity, then this entity is, almost by definition, the single member of the last conceptual level. It is of course possible to imagine still lower levels, to imagine this single entity as being built from several ingredients; but as long as all physical phenomena can be expressed in terms of the single entity considered as a unit, the several ingredients will always appear in exactly the same proportions, in exactly the same combination. Such an unvarying combination of the several ingredients might just as well be given a single name and treated as a unit, since no physical operation will be able to separate them. A last conceptual level, containing onlv a single member, will here be postulated. The single member will be called the primitive field and will be denoted by the symbol ξ:

(y) Elementary particles: n, p, e, ...

(z) Primitive field: ξ

It is postulated, as the basis of this theory, that there exists a primitive wave field, ξ, out of which all the elementary particles are to be built. The theoretical development will consist of the building of various structures from such a primitive wave field and the attempt to identify among these structures certain ones which have properties similar to the observed properties of known elementary particles.

THE PRIMITIVE FIELD

Most of the properties of a single primitive wave field can be inferred from its singleness. Because some of the elementary particles are known to be Fermi-Dirac particles, it is essential that the primitive field have Fermi-Dirac statistics. A Fermi-Dirac structure cannot possibly be built from any number of Bose-Einstein building blocks, whereas both Fermi-Dirac and Bose-Einstein structures can be formed, respectively, from odd and even numbers of Fermi-Dirac building units. Similarly, since there are known to be elementary particles with a spin of

one-half, the primitive field cannot have integral spin but must have half-integral spin; for simplicity a spin of one-half was tentatively selected for the primitive field. Because the photon and neutrino are included in the list of elementary particles to be constructed from the primitive field, it was inferred that the primitive waves should have a velocity essentially equal to c, and negligible mass.

Any coupling between primitive waves must be of a very restricted kind. The usual general forms of coupling would imply coupling fields, with their own quanta, and these quanta would be distinguishable from ξ yet equally fundamental. That is, there would then be two or more members of the lowest conceptual level, and this was ruled out at the start. It will be assumed tentatively that the only coupling between primitive waves is via the exclusion principle. (But see also Appendix D.)

A wave equation for a primitive wave, embodying the properties listed above, can be written down directly. The insertion of zero mass into the Dirac equation gives the equation of a wave with velocity c and spin one-half. The question of statistics and coupling will not arise until two or more waves are combined, as in the next section.

In order to simplify future calculations, Dirac's representation of the matrix operators in his equation is modified by introduction of the diagonal matrix τ to replace Dirac's matrix ρ_1 . As a result, the four-component wave function can be separated into a pair of two-component spinors which are not coupled by the wave equation. All calculations can then be carried out in terms of the two-component spinors, and this is a great advantage when two or more waves are combined into a single structure. For a single primitive wave, the explicit wave equation will be written in the following form:

$$\left[\frac{1}{c}\frac{\partial}{\partial t} + \tau\vec{\sigma}\cdot\nabla\right]\xi(x,y,z,t) = 0 \qquad (1)$$

Equation (1) involves four matrix operators: τ, σ_x , σ_y , σ_z . These operators can be written out in full four-by-four notation, with dots representing zeros:

$$\tau = \boxed{\begin{matrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{matrix}} \qquad (2a)$$

$$\sigma_x = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \qquad (2b)$$

$$\sigma_y = \begin{bmatrix} \cdot & -i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -i \\ \cdot & \cdot & i & \cdot \end{bmatrix} \qquad (2c)$$

$$\sigma_z = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix} \qquad (2d)$$

With the above notation, the wave function $\xi(x,y,z,t)$ becomes a column matrix of four components. However, the matrices (2) have no elements coupling the upper two components with the lower two components. The upper and lower pairs of components may therefore be treated separately, in Equation (1), with τ taking on the

respective values +1 and -1, while the components of $\vec{\sigma}$ have been reduced to their two-by-two forms:

$$\sigma_x = \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix}^{\sigma} \qquad \sigma_y = \begin{bmatrix} \cdot & -i \\ i & \cdot \end{bmatrix}^{\sigma} \qquad \sigma_z = \begin{bmatrix} 1 & \cdot \\ \cdot & -1 \end{bmatrix}^{\sigma} \tag{3}$$

In practice it will be found convenient to separate the τ- and σ-dependence of the wave function; this procedure will become clearer in later sections.

PLANE WAVE SOLUTION

Under certain conditions, Equation (1) can be satisfied by a plane wave solution of the form:

$$\xi(\vec{r},t;\vec{\kappa},\kappa_0) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot e^{i(\vec{\kappa}\cdot\vec{r} - \kappa_0 ct)} \tag{4}$$

where a,b,c,d are constants independent of x,y,z,t.

Substitution of (4) and (2) into (1) leads to the following set of equations, linear and homogeneous in the unknown parameters a,b,c,d.

$$\left.\begin{aligned} (K_0 - K_z)\,a - (K_x - iK_y)\,b &= 0 \\ -(K_x + iK_y)\,a + (K_0 + K_z)\,b &= 0 \\ (K_0 + K_z)\,c + (K_x - iK_y)\,d &= 0 \\ (K_x + iK_y)\,c + (K_0 - K_z)\,d &= 0 \end{aligned}\right\} \qquad (5)$$

Either the first or the second equation in (5) can be solved for the ratio a/b, but these two equations will give consistent results only if the following secular relationship is satisfied:

$$K_x^2 + K_y^2 + K_z^2 = K_0^2 \qquad (6)$$

This same relationship is also required, in order that the third and fourth equations of (5) give a consistent value for the ratio c/d. In each case, however, only the ratio is determined. To fix the individual parameters, except for an adjustable phase factor, the following normalization requirement will be imposed:

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = |K_0| \tag{7}$$

It has already been noted that the representation chosen for the operators is such that the first pair of components is quite independent of the second pair. There are two independent solutions of Equations (5), each satisfying both (6) and (7), and these two solutions are given below in (8) and (9):

$$\left.\begin{aligned} a' &= \left[\frac{(K_0 + K_z)(K_x - iK_y)}{2\,(K_x^2 + K_y^2)^{1/2}}\right]^{1/2} \\ b' &= \left[\frac{(K_0 - K_z)(K_x + iK_y)}{2\,(K_x^2 + K_y^2)^{1/2}}\right]^{1/2} \\ c' &= d' = 0 \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} a'' &= b'' = 0 \\ c'' &= +\left[\frac{(K_0 - K_z)(K_x - iK_y)}{2\,(K_x^2 + K_y^2)^{1/2}}\right]^{1/2} \\ d'' &= -\left[\frac{(K_0 + K_z)(K_x + iK_y)}{2\,(K_x^2 + K_y^2)^{1/2}}\right]^{1/2} \end{aligned}\right\} \tag{9}$$

These two independent solutions may conveniently be written in a factored notation, which can be defined as follows:

$$\left.\begin{aligned} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &\equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\tau} \begin{bmatrix} a \\ b \end{bmatrix}^{\sigma} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\tau} \begin{bmatrix} c \\ d \end{bmatrix}^{\sigma} \\ &\equiv \begin{bmatrix} a \\ c \end{bmatrix}^{\tau} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\sigma} + \begin{bmatrix} b \\ d \end{bmatrix}^{\tau} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\sigma} \end{aligned}\right\} \qquad (10)$$

In the above factored notation, the σ-operators take the form given in Equations (3), and operate upon the two-component spinors labeled with the superscript σ, while the spinors labeled τ are acted on by the operator τ, which is here represented by a two-by-two matrix:

$$\tau = \begin{bmatrix} 1 & \cdot \\ \cdot & -1 \end{bmatrix}^{\tau} \qquad (11)$$

In the factored notation, the two independent plane wave solutions are:

$$\xi' = \boxed{\begin{matrix} 1 \\ 0 \end{matrix}}^{\tau} \cdot \boxed{\begin{matrix} a' \\ b' \end{matrix}}^{\sigma} \cdot e^{i(\vec{\kappa}\cdot\vec{r} - \kappa_0 ct) + i\phi'} \qquad (12a)$$

$$\xi'' = \boxed{\begin{matrix} 0 \\ 1 \end{matrix}}^{\tau} \cdot \boxed{\begin{matrix} c'' \\ d'' \end{matrix}}^{\sigma} \cdot e^{i(\vec{\kappa}\cdot\vec{r} - \kappa_0 ct) + i\phi''} \qquad (12b)$$

in which the constants a',b',c",d" have the values given in (8) and (9), while the phases ϕ' and ϕ'' are real numbers but are otherwise arbitrary.

DOUBLE-WAVE EQUATION

Except for changes in normalization and matrix representation, the equations for the primitive field have so far been taken from the formalism of the Dirac theory, the wave equation (1) containing a differential operator acting on a wave function. The Schrödinger equation has a similar general form, and can indeed be derived from the Dirac equation. In generalizing the present theory to cover the description of systems of two or more primitive waves, the formal procedure will be borrowed from the Schrödinger theory. That is, wave functions for individual waves will be multiplied together to give a wave function for the composite system, and the corresponding differential operators will be added together. When applied to the Dirac theory, this is often called the "many-time formalism", but in the present theory the simplicity of the coupling between waves makes it convenient to write most of the equations in terms of a single "laboratory time". It should be noted at this point that the choice of a particular scheme for describing wave systems is a tentative one.

Plane wave solutions of the single-wave equation (1) have already been described and are given in (4) and (12). A more general solution to Equation (1) can be formed from a linear combination of plane wave solutions. Two such linear combinations might be written in this way:

$$\xi_A = \int A(\vec{K}_a, K_{oa}) \cdot \xi(\vec{r}_1, t_1; \vec{K}_a, K_{oa}) \cdot dK_{xa}\, dK_{ya}\, dK_{za} \qquad (13\,a)$$

$$\xi_B = \int B(\vec{K}_b, K_{ob}) \cdot \xi(\vec{r}_2, t_2; \vec{K}_b, K_{ob}) \cdot dK_{xb}\, dK_{yb}\, dK_{zb} \qquad (13\,b)$$

The labels 1 and 2 serve to identify the two solutions. A general wave function describing the combined system built from ξ_A and ξ_B, and antisymmetrized to show the Fermi-Dirac nature of the primitive field, will be:

$$\begin{aligned}\Psi(1,2) = \iint & A(\vec{K}_a, K_{oa}) \cdot B(\vec{K}_b, K_{ob}) \cdot \\ & \cdot \Big\{ \xi(\vec{r}_1, t_1; \vec{K}_a, K_{oa}) \cdot \xi(\vec{r}_2, t_2; \vec{K}_b, K_{ob}) \\ & \quad - \xi(\vec{r}_2, t_2; \vec{K}_a, K_{oa}) \cdot \xi(\vec{r}_1, t_1; \vec{K}_b, K_{ob}) \Big\} \cdot \\ & \cdot dK_{xa}\, dK_{ya}\, dK_{za}\, dK_{xb}\, dK_{yb}\, dK_{zb} \end{aligned} \qquad (14)$$

It is important to distinguish between the adding of solutions to produce a new single-wave solution, as in (13a), and the multiplication of solutions to produce a multiple-wave system, as in (14). While this distinction does follow current practice in quantum mechanics and field theory, it should be considered tentative here, until it can be proved from the initial postulate or justified by the results of the theory.

The double-wave function $\Psi(1,2)$ will separately satisfy both of the following single-wave equations:

$$\left[\frac{1}{c}\frac{\partial}{\partial t_1} + \tau_1 \vec{\sigma}_1 \cdot \nabla_1\right] \Psi(1,2) = 0 \qquad (15a)$$

$$\left[\frac{1}{c}\frac{\partial}{\partial t_2} + \tau_2 \vec{\sigma}_2 \cdot \nabla_2\right] \Psi(1,2) = 0 \qquad (15b)$$

It is at this point that the separate times, t_1 and t_2, can be replaced by a center-of-gravity or laboratory time, T, and a relative time, t_r. The separate space coordinates, $\vec{r}_1$ and $\vec{r}_2$, can at the same time be replaced by center-of-gravity and relative coordinates, $\vec{R}$ and $\vec{r}$.

The new variables are related to the old by equations which include the following:

$$
\left.\begin{aligned}
T &= \tfrac{1}{2}(t_1 + t_2) & t_r &= t_1 - t_2 \\
\vec{R} &= \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) & \vec{r} &= \vec{r}_1 - \vec{r}_2 \\
\frac{\partial}{\partial T} &= \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} & \frac{\partial}{\partial t_r} &= \frac{1}{2}\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right) \\
\nabla_1 &= \tfrac{1}{2}\nabla_R + \nabla_r & \nabla_2 &= \tfrac{1}{2}\nabla_R - \nabla_r
\end{aligned}\right\} \qquad (16)
$$

From Equations (15a) and (15b), by addition and subtraction, the following equations may be derived:

$$
\left[\frac{1}{c}\frac{\partial}{\partial T} + \frac{1}{2}(\tau_1\vec{\sigma}_1 + \tau_2\vec{\sigma}_2)\cdot\nabla_R + (\tau_1\vec{\sigma}_1 - \tau_2\vec{\sigma}_2)\cdot\nabla_r\right]\Psi(1,2) = 0 \qquad (17)
$$

$$
\left[\frac{1}{c}\frac{\partial}{\partial t_r} + \frac{1}{4}(\tau_1\vec{\sigma}_1 - \tau_2\vec{\sigma}_2)\cdot\nabla_R + \frac{1}{2}(\tau_1\vec{\sigma}_1 + \tau_2\vec{\sigma}_2)\cdot\nabla_r\right]\Psi(1,2) = 0 \qquad (18)
$$

Equation (17) has the same general form as the conventional wave equation for a field of a certain mass (M) and any spin:

$$\left[\beta_\mu \frac{\partial}{\partial x_\mu} + \frac{Mc}{\hbar}\right] \Phi = 0 \qquad (19)$$

In (19) the repeated index μ indicates a summation over the four space-time coordinates of relativity theory. The β_μ are operators whose commutation rules depend on the spin of the field Φ. M is the mass of the field. A comparison of terms in (19) with terms in (17) shows that the double-wave system $\Psi(1,2)$ satisfies the same law of motion as the field Φ, except that the mass of the field Φ is replaced by an operator which acts on variables in $\Psi(1,2)$ which are concerned with the inner structure of the double-wave system. That is, the double-wave system as a whole may be treated as a field or particle, with the internal structure or relative motion, as selected by the operator $(\tau_1\vec{\sigma}_1 - \tau_2\vec{\sigma}_2)\cdot\nabla_r$, accounting for the effects which we know by the name of <u>mass</u>. This is a tentative <u>mathematical</u> suggestion based on the similarity between (19) and (17), but its <u>physical</u> interpretation has the attraction of great

simplicity. It is known that an increase in the kinetic energy of internal motion (as, for example, the spinning of a gyroscope) will be reflected in a finite increase in the mass of the system. Thus the mass of a structure will contain contributions due to known internal motion; it would be gratifying if all mass could be shown to arise in the same way, from some form of internal motion.

SPIN ZERO SOLUTION

The wave function $\Psi(1,2)$ will be separated into two factors, representing the center-of-gravity motion and the internal structure:

$$\Psi(1,2) = e^{i\vec{K}\cdot\vec{R} - iK_0 cT} \cdot \psi(\vec{r}) \tag{20}$$

In (20) the relative time, t_r, was omitted from explicit consideration, since it will not be involved in the solution of the wave equation (17). Strictly, (20) can be considered as a part of a more general solution, or as a special case in which the relative time, t_r, has been set equal to zero.

Substitution of (20) into (17) gives:

$$K_0\,\psi(\vec{r}) = \Big[\tfrac{1}{2}(\tau_1\vec{\sigma_1} + \tau_2\vec{\sigma_2})\cdot\vec{K} + (\tau_1\vec{\sigma_1} - \tau_2\vec{\sigma_2})\cdot\tfrac{1}{i}\nabla_r\Big]\,\psi(\vec{r}) \qquad (21)$$

Since both τ_1 and τ_2 are diagonal, and each commutes with everything in (21), they may each be given the numerical values of ± 1. For the first solution to be examined, both τ_1 and τ_2 will be set equal to +1.

$$K_0\,\psi(\vec{r}) = \Big[\tfrac{1}{2}(\vec{\sigma_1} + \vec{\sigma_2})\cdot\vec{K} + (\vec{\sigma_1} - \vec{\sigma_2})\cdot\tfrac{1}{i}\nabla_r\Big]\,\psi(\vec{r}) \qquad (22)$$

A further simplification can come from the selection of a solution for which the center of gravity is at rest, so that $\vec{K}$ is equal to zero:

$$K_0\,\psi(\vec{r}) = (\vec{\sigma_1} - \vec{\sigma_2})\cdot\tfrac{1}{i}\nabla_r\,\psi(\vec{r}) \qquad (23)$$

Equation (23) is now an eigenfunction-eigenvalue problem that can be solved directly.

The form of Equation (23) is such that a solution $\psi(\vec{r})$ can be restricted to have a definite value of total internal angular momentum. The value zero will be chosen, although other integral values could also be studied.

It will be necessary to have a definite spin notation, to show the various σ-components and τ-components which are included in $\psi(\vec{r})$. Each of the two waves which make up the double-wave structure can have two choices for its σ-spin, and two choices for its τ-spin, although in the present case, with Equation (23), the τ-components have already been selected and fixed. For representing either kind of spin, the following basic frame will be used:

$$\begin{array}{|c|}
\hline
\chi^{+}(1)\cdot\chi^{+}(2) \\
\hline
\chi^{+}(1)\cdot\chi^{-}(2) \\
\chi^{-}(1)\cdot\chi^{+}(2) \\
\hline
\chi^{-}(1)\cdot\chi^{-}(2) \\
\hline
\end{array} \qquad (24)$$

For example, a state in which both τ_1 and τ_2 have the same value of +1 can be indicated by the notation:

$$ {}^3(\tau)^+ \equiv \left[\begin{array}{|c|}\hline 1 \\ \hline \cdot \\ \cdot \\ \hline \cdot \\ \hline \end{array}\right]^{\tau} = \chi^+(\tau_1)\cdot\chi^+(\tau_2) \tag{25} $$

A singlet combination of the two σ-spins can be written:

$$ {}^1(1)_0 \equiv \frac{1}{2^{1/2}}\cdot\left[\begin{array}{|c|}\hline \cdot \\ \hline 1 \\ -1 \\ \hline \cdot \\ \hline \end{array}\right]^{\sigma} = \frac{1}{2^{1/2}}\cdot\left[\chi^+(\sigma_1)\cdot\chi^-(\sigma_2) - \chi^-(\sigma_1)\cdot\chi^+(\sigma_2)\right] \tag{26} $$

The brief notation on the left of (26) will serve as a form of abbreviation which can be generalized to other functions of the spinor components. In addition to the $^1S_0$ function given in (26), there is a $^3P_0$ function which can be formed from the vector $\vec{r}$ and the available spin components. This $^3P_0$ function is given in (27):

$$
{}^{3}(\vec{r})_{0} \equiv \frac{1}{2^{1/2}} \begin{array}{|c|}\hline -(x-iy) \\ \hline z \\ z \\ \hline (x+iy) \\ \hline \end{array}^{\,\sigma} \qquad (27)
$$

The two σ-spin functions, (26) and (27), are the only ones which satisfy the requirement that the total internal angular momentum vanish (J = 0), under the condition that $\vec{K}$ be zero. Later, when the motion of the center of gravity is considered, so that $\vec{K}$ differs from zero, two more σ-spin functions will be introduced. But when the center of gravity is at rest, it is sufficient to write $\psi(\vec{r})$ as the sum of two terms:

$$
\psi(\vec{r}) = a \cdot \psi_a + b \cdot \psi_b \qquad (28)
$$

$$
\psi_a = {}^{3}(\tau)^{+} \cdot {}^{1}(1)_{0} \cdot f_a(r) \qquad (29a)
$$

$$
\psi_b = {}^{3}(\tau)^{+} \cdot {}^{3}(\vec{r})_{0} \cdot f_b(r) \qquad (29b)
$$

It should be noted that both spin functions (26) and (27) are antisymmetric with respect to exchange of the labels 1 and 2. It is required that $\psi(\vec{r})$ be antisymmetric. Thus the scalar functions $f_a(r)$ and $f_b(r)$ should be symmetric, and will actually be found to be functions of r^2.

The general solution (28) can now be substituted into the wave equation (23). In the spinor notation of (24), the operator $(\vec{\sigma_1} - \vec{\sigma_2})\cdot\nabla_r$ has the form:

$$(\vec{\sigma_1} - \vec{\sigma_2})\cdot\nabla_r = \left[\begin{array}{c|cc|c} \cdot & -\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) & \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) & \cdot \\ \hline -\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) & 2\frac{\partial}{\partial z} & \cdot & \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) & \cdot & -2\frac{\partial}{\partial z} & -\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \hline \cdot & \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) & -\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) & \cdot \end{array}\right]^{\sigma} \tag{30}$$

Equations (23), (26-30) can be used to obtain a pair of second-order differential equations for the unknown scalar functions $f_a(r)$ and $f_b(r)$:

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \kappa^2\right] f_a(r) = 0 \tag{31a}$$

$$\left[\frac{d^2}{dr^2} + \frac{4}{r}\frac{d}{dr} + \kappa^2\right] f_b(r) = 0 \tag{31b}$$

where an abbreviation has been used:

$$\kappa^2 = \left(\frac{K_0}{2}\right)^2 \tag{32}$$

That solution of Equation (31a) which is finite at the origin, r = 0, is the spherical bessel function $j_0(\kappa r)$:

$$j_0(\kappa r) = \frac{\sin(\kappa r)}{\kappa r} \qquad (33a)$$

Similarly, the solution of (31b) finite at the origin is also a spherical bessel function:

$$j_1(\kappa r) = \frac{\sin(\kappa r) - (\kappa r)\cos(\kappa r)}{(\kappa r)^3} \qquad (33b)$$

These spherical bessel functions satisfy the general relations:

$$\frac{d}{dr} j_n(\kappa r) = -\kappa^2 r \cdot j_{n+1}(\kappa r) \qquad (34)$$

$$j_n(\kappa r) = \frac{1}{2n+1}\left[j_{n-1}(\kappa r) + \kappa^2 r^2 \cdot j_{n+1}(\kappa r)\right] \qquad (35)$$

For convenience the solutions of Equations (31a) and (31b) will be written in the following way:

$$f_a(r) = j_0(\kappa r) \qquad (36a)$$

$$f_b(r) = i\kappa\, j_1(\kappa r) \qquad (36b)$$

Then the solution of Equation (23) can be written:

$$K_0(\psi_a \pm \psi_b) = (\sigma_1 - \sigma_2)\cdot\frac{1}{i}\nabla_r\,(\psi_a \pm \psi_b) \qquad (37)$$
$$= \pm 2\kappa\,(\psi_a \pm \psi_b)$$

That is, the operator $(\vec{\sigma}_1 - \vec{\sigma}_2)\cdot\frac{1}{i}\nabla_r$ has the eigenvalues $+2\kappa$ and -2κ, which belong to the eigenfunctions $(\psi_a + \psi_b)$ and $(\psi_a - \psi_b)$, respectively. These eigenvalues are numerical values for the quantity K_0, which represents the energy or frequency of the double-wave structure, in wavenumber units. Since this is the rest system, 2κ represents the mass or rest-frequency of the structure, again in wavenumber units. It remains to be verified, in later sections, that this structure really moves like a particle-with-mass, when the center of gravity is allowed to move through space.

There is also a question as to the significance of the plus-or-minus signs in (37). This question cannot be answered at this stage of the theoretical development, but it can be pointed out that there would have been a sign reversal if, in Equation (21), τ_1 and τ_2 had been put equal to -1 instead of +1. The appropriate τ-function would then have been:

$${}^{3}(\tau)^{-} \equiv \begin{array}{|c|}\hline \cdot \\ \hline \cdot \\ \cdot \\ \hline 1 \\ \hline \end{array}^{\,\tau} \qquad (38)$$

Instead of (28) and (29), the expansion of $\psi(\vec{r})$ would have been:

$$\psi(\vec{r}) = a\,\psi_a' + b\,\psi_b' \qquad (39)$$

$$\psi_a' = {}^{3}(\tau)^{-}\ {}^{1}(1)_{0}\ f_a(r) \qquad (40a)$$

$$\psi_b' = {}^{3}(\tau)^{-}\ {}^{3}(\vec{r})_{0}\ f_b(r) \qquad (40b)$$

Equations (30-36) would have remained applicable, but instead of Equation (37) the solution would have been:

$$K_0\left(\psi_a' \pm \psi_b'\right) = \mp 2K\cdot\left(\psi_a' \pm \psi_b'\right) \qquad (41)$$

with a reversal in the signs attached to the quantity 2K. Even though, for other reasons, this particular structure will be found to be an unsatisfactory model of an elementary particle, it is nevertheless permissible to anticipate that the existence of both positive and negative solutions for the rest frequency K_0 may have some correlation with the existence in nature of certain elementary particles along with their associated antiparticles.

THE RELATIVE TIME

Equation (20) can be generalized to allow for a dependence upon the relative time variable, t_r .

$$\Psi(1,2) = e^{i\vec{K}\cdot\vec{R} - iK_0 cT} \cdot \phi(\vec{r}, t_r) \qquad (42)$$

The new function $\phi(\vec{r}, t_r)$ can be related to $\psi(\vec{r})$:

$$\phi(\vec{r}, 0) = \psi(\vec{r}) \qquad (43)$$

When (42) is substituted into the relative time equation (18), the result, analogous to (21), is:

$$\frac{i}{c}\frac{\partial}{\partial t_r}\phi(\vec{r}, t_r) = \Big[\tfrac{1}{4}(\tau_1\vec{\sigma}_1 - \tau_2\vec{\sigma}_2)\cdot\vec{K} + \tfrac{1}{2}(\tau_1\vec{\sigma}_1 + \tau_2\vec{\sigma}_2)\cdot\tfrac{1}{i}\nabla_r\Big]\phi(\vec{r}, t_r) \qquad (44)$$

As before, τ_1 and τ_2 will be set equal to +1:

$$\frac{i}{c}\frac{\partial}{\partial t_r}\phi(\vec{r}, t_r) = \Big[\tfrac{1}{4}(\vec{\sigma}_1 - \vec{\sigma}_2)\cdot\vec{K} + \tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\tfrac{1}{i}\nabla_r\Big]\phi(\vec{r}, t_r) \qquad (45)$$

With the center of gravity at rest, $\vec{K} = 0$:

$$\frac{i}{c}\frac{\partial}{\partial t_r}\phi(\vec{r},t_r) = \frac{1}{2}(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\frac{1}{i}\nabla_r\,\phi(\vec{r},t_r) \qquad (46)$$

Equation (46) can now be solved, with the aid of (43) and the following easily verified relations:

$$(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\nabla_r\,\psi_a = 0 \qquad (47a)$$

$$(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\nabla_r\,\psi_b = 0 \qquad (47b)$$

Here ψ_a and ψ_b are as given in (29), and the operator $(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\nabla_r$ has the form:

$$(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\nabla_r = \begin{array}{c} \hfill \sigma \\ \begin{bmatrix} 2\frac{\partial}{\partial z} & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) & \cdot \\ \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & \cdot & \cdot & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \\ \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & \cdot & \cdot & \left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right) \\ \cdot & \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & \left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) & -2\frac{\partial}{\partial z} \end{bmatrix} \end{array} \qquad (48)$$

It follows, from Equations (28), (29), (43), (46), and (47), that, in this special case with $\vec{K} = 0$,

$$\left[\frac{i}{c}\frac{\partial}{\partial t_r}\phi(\vec{r},t_r)\right]_{t_r=0} = 0 \tag{49}$$

Furthermore, repeated differentiation of (46) with respect to the relative time, t_r , gives, for any value of n:

$$\left[\frac{\partial^n}{\partial t_r^{\,n}}\phi(\vec{r},t_r)\right]_{t_r=0} = 0 \tag{50}$$

As long as $\vec{K} = 0$, $\phi(\vec{r},t_r)$ can be taken as constant with respect to variations of the relative time, t_r , so that $\psi(\vec{r})$, as given in (28), (29), (36), (37), is a satisfactory solution of Equation (46).

When $\vec{K}$ differs from zero, Equations (22) and (45) must be used instead of (23) and (46), and the question of the wave function's dependence upon the relative time, t_r , will then have to be re-examined. In Appendix B such a re-examination is carried out.

MOTION OF CENTER OF GRAVITY

When the center of gravity is in motion, so that $\vec{K} \neq 0$, the more general wave equation (22) must be used instead of the simpler equation (23). The wave function $\psi(\vec{r})$ will also be more complicated. Instead of the two terms (29), an infinite number of terms will be used in the expansion of $\psi(\vec{r})$, but these will be arranged in increasing powers of $\vec{K}$; that is, in order of increasingly complex dependence upon the components of $\vec{K}$. If ψ_a and ψ_b in (29) are taken as the leading terms in this series, then the rest of the terms are generated from ψ_a and ψ_b by the operators that appear on the right-hand side of the wave equation (22).

It should be pointed out that such a procedure for generating the terms in the expansion of $\psi(\vec{r})$ assumes that the wavenumber $\varkappa$ can play the part of a parameter independent of $\vec{K}$, while K_0 is allowed to vary with $\vec{K}$. Thus Equation (32), through which $\varkappa$ was originally introduced, will hold only when $\vec{K} = 0$, and will need to be replaced by a more general equation, to be found from the solution of (22).

It will be convenient, because of the complexity of the expansion, to divide the wave equation (22) by the parameter $\varkappa$, and to introduce the following abbreviations:

$$w \equiv \frac{K_e}{\varkappa} \tag{51a}$$

$$\vec{k} \equiv \frac{\vec{K}}{\varkappa} \tag{51b}$$

The wave equation can then be written:

$$w\,\psi(\vec{r}) = \left[\tfrac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\vec{k} + (\vec{\sigma}_1 - \vec{\sigma}_2)\cdot\frac{1}{i\varkappa}\nabla_r\right]\psi(\vec{r}) \tag{52}$$

In addition to the previous σ-spin functions, ${}^1(1)_0$ and ${}^3(\vec{r})_0$, in (26) and (27), there will be two new σ-spin functions:

$${}^3(\vec{k})_0 \equiv \frac{1}{2^{1/2}}\cdot \begin{array}{|c|}\hline -(k_x - i k_y) \\ \hline k_z \\ k_z \\ \hline (k_x + i k_y) \\ \hline \end{array}^{\,\sigma} \tag{53}$$

$$
{}^3(\vec{k}\times\vec{r})_0 \equiv \frac{1}{2^{k}} \cdot \left.\begin{array}{|c|}
\hline
-\left[(k_y z - k_z y) - i(k_z x - k_x z)\right] \\
\hline
(k_x y - k_y x) \\
(k_x y - k_y x) \\
\hline
\left[(k_y z - k_z y) + i(k_z x - k_x z)\right] \\
\hline
\end{array}\right.^{\sigma} \qquad (54a)
$$

The above function (54a) can also be written in the form:

$$
{}^3(i\vec{k}\times\vec{r})_0 \equiv \frac{1}{2^{k}} \cdot \left.\begin{array}{|c|}
\hline
(k_x - ik_y)z - k_z(x - iy) \\
\hline
i(k_x y - k_y x) \\
i(k_x y - k_y x) \\
\hline
(k_x + ik_y)z - k_z(x + iy) \\
\hline
\end{array}\right.^{\sigma} \qquad (54b)
$$

A number of identities, useful in the solution of equation (52) and the analogous relative-time equation, will be written down here:

$$
\nabla_r\, j_n(\kappa r) = -\kappa^2 \cdot j_{n+1}(\kappa r)\cdot \vec{r} \qquad (55)
$$

$$
\left.\begin{aligned}
(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\nabla_r\, {}^3(\vec{r})_0 &= 0 \\
(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\nabla_r\, {}^3(i\vec{k}\times\vec{r})_0 &= -4\cdot {}^3(\vec{k})_0
\end{aligned}\right\} \qquad (56)
$$

$$
\left.\begin{aligned}
(\vec{\sigma}_1-\vec{\sigma}_2)\cdot\nabla_r\, {}^3(\vec{r})_0 &= 6\cdot {}^1(1)_0 \\
(\vec{\sigma}_1-\vec{\sigma}_2)\cdot\nabla_r\, {}^3(i\vec{k}\times\vec{r})_0 &= 0
\end{aligned}\right\} \qquad (57)
$$

$$\left.\begin{aligned}
(\vec\sigma_1+\vec\sigma_2)\cdot\vec k\ {}^1(1)_0 &= 0\\
(\vec\sigma_1+\vec\sigma_2)\cdot\vec k\ {}^3(\vec r)_0 &= 2\,{}^3(i\vec k\times\vec r)_0\\
(\vec\sigma_1+\vec\sigma_2)\cdot\vec k\ {}^3(\vec k)_0 &= 0\\
(\vec\sigma_1+\vec\sigma_2)\cdot\vec k\ {}^3(i\vec k\times\vec r)_0 &= 2\,k^2\cdot{}^3(\vec r)_0 - 2\,(\vec k\cdot\vec r)\cdot{}^3(k)_0
\end{aligned}\right\} \quad (58)$$

$$\left.\begin{aligned}
(\vec\sigma_1+\vec\sigma_2)\cdot\vec r\ {}^1(1)_0 &= 0\\
(\vec\sigma_1+\vec\sigma_2)\cdot\vec r\ {}^3(\vec r)_0 &= 0\\
(\vec\sigma_1+\vec\sigma_2)\cdot\vec r\ {}^3(\vec k)_0 &= -2\,{}^3(i\vec k\times\vec r)\\
(\vec\sigma_1+\vec\sigma_2)\cdot\vec r\ {}^3(i\vec k\times\vec r)_0 &= 2\,(\vec k\cdot\vec r)\cdot{}^3(\vec r)_0 - 2\,r^2\cdot{}^3(k)_0
\end{aligned}\right\} \quad (59)$$

$$\left.\begin{aligned}
(\vec\sigma_1-\vec\sigma_2)\cdot\vec k\ {}^1(1)_0 &= 2\,{}^3(\vec k)_0\\
(\vec\sigma_1-\vec\sigma_2)\cdot\vec k\ {}^3(\vec r)_0 &= 2\,(\vec k\cdot\vec r)\ {}^1(1)_0\\
(\vec\sigma_1-\vec\sigma_2)\cdot\vec k\ {}^3(\vec k)_0 &= 2\,k^2\ {}^1(1)_0\\
(\vec\sigma_1-\vec\sigma_2)\cdot\vec k\ {}^3(i\vec k\times\vec r)_0 &= 0
\end{aligned}\right\} \quad (60)$$

$$\left.\begin{aligned}
(\vec\sigma_1-\vec\sigma_2)\cdot\vec r\ {}^1(1)_0 &= 2\,{}^3(\vec r)_0\\
(\vec\sigma_1-\vec\sigma_2)\cdot\vec r\ {}^3(\vec r)_0 &= 2\,r^2\ {}^1(1)_0\\
(\vec\sigma_1-\vec\sigma_2)\cdot\vec r\ {}^3(\vec k)_0 &= 2\,(\vec k\cdot\vec r)\ {}^1(1)_0\\
(\vec\sigma_1-\vec\sigma_2)\cdot\vec r\ {}^3(i\vec k\times\vec r)_0 &= 0
\end{aligned}\right\} \quad (61)$$

In the wave equation (52), the operator on the right hand side will be denoted by the symbol H:

$$H = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)\cdot\vec{k} + (\vec{\sigma}_1 - \vec{\sigma}_2)\cdot\frac{1}{iK}\nabla_r \tag{62}$$

This H is equivalent to the usual Hamiltonian divided by the product $\hbar\kappa$. The wave function $\psi(\vec{r})$ will be expanded in a series:

$$\psi(\vec{r}) = a\cdot\psi_a + b\cdot\psi_b + c\cdot\psi_c + d\cdot\psi_d + \cdots \tag{63}$$

The coefficients a,b,c,..., will be determined later through the solution of a set of simultaneous equations, but first it will be necessary to identify the separate functions, the ψ_a, ψ_b, ψ_c, and so forth, and to calculate the matrix elements of the operator H.

As mentioned earlier, the procedure will be to start with the ψ_a and ψ_b already given in (29) and (36), and to operate upon these two, and on each successive function when it has been identified, with the operator H given in (62). In this way the first fifteen functions of the infinite set have been determined. The list, (64), omits the common factor ${}^3(\tau)^+$, which should be understood in each case.

$$\psi_a = j_0(\kappa r)\cdot {}^1(1)_0 \qquad (64a - 64j)$$

$$\psi_b = i\kappa j_1 \cdot {}^3(\vec r)_0$$

$$\psi_c = i\kappa j_1 \cdot {}^3(i\,\vec k\times\vec r)_0$$

$$\psi_d = -i\kappa j_1 \cdot\left\{(\vec k\cdot\vec r)\cdot {}^3(\vec k)_0 - \tfrac{1}{3}k^2\cdot {}^3(\vec r)_0\right\}$$

$$\psi_e = \kappa^2 j_2 \cdot\left\{(\vec k\cdot\vec r)^2 - \tfrac{1}{3}k^2 r^2\right\}\cdot {}^1(1)_0$$

$$\psi_f = i\kappa^3 j_3\cdot\left\{\left[(\vec k\cdot\vec r)^2 - \tfrac{1}{5}k^2 r^2\right]\cdot {}^3(\vec r)_0 - \tfrac{2}{5}(\vec k\cdot\vec r)\,r^2\cdot {}^3(\vec k)_0\right\}$$

$$\psi_g = i\kappa^3 j_3\cdot\left\{(\vec k\cdot\vec r)^2 - \tfrac{1}{5}k^2 r^2\right\}\cdot {}^3(i\,\vec k\times\vec r)_0$$

$$\psi_h = -i\kappa^3 j_3\cdot\left\{\left[(\vec k\cdot\vec r)^3 - \tfrac{3}{7}(\vec k\cdot\vec r)k^2 r^2\right]\cdot {}^3(\vec k)_0 + \left[-\tfrac{3}{7}(\vec k\cdot\vec r)^2 k^2 + \tfrac{3}{35}k^4 r^2\right]\cdot {}^3(\vec r)_0\right\}$$

$$\psi_i = \kappa^4 j_4\cdot\left\{(\vec k\cdot\vec r)^4 - \tfrac{6}{7}(\vec k\cdot\vec r)^2 k^2 r^2 + \tfrac{3}{35}k^4 r^4\right\}\cdot {}^1(1)_0$$

$$\psi_j = i\kappa^5 j_5\cdot\left\{\left[(\vec k\cdot\vec r)^4 - \tfrac{2}{3}(\vec k\cdot\vec r)^2 k^2 r^2 + \tfrac{1}{21}k^4 r^4\right]\cdot {}^3(\vec r)_0 + \left[-\tfrac{4}{9}(\vec k\cdot\vec r)^3 r^2 + \tfrac{4}{21}(\vec k\cdot\vec r)k^2 r^4\right]\cdot {}^3(\vec k)_0\right\}$$

$$\psi_k = i\varkappa^5 j_5 \cdot \left\{ (\vec{k}\cdot\vec{r})^4 - \frac{2}{3}(\vec{k}\cdot\vec{r})^2 k^2 r^2 + \frac{1}{21} k^4 r^4 \right\} \cdot {}^3(i\vec{k}\times\vec{r})_0$$

$$\psi_l = -i\varkappa^5 j_5 \cdot \left\{ \left[(\vec{k}\cdot\vec{r})^5 - \frac{10}{11}(\vec{k}\cdot\vec{r})^3 k^2 r^2 + \frac{5}{33}(\vec{k}\cdot\vec{r}) k^4 r^4 \right] \cdot {}^3(\vec{k})_0 \right.$$
$$\left. + \left[-\frac{5}{11}(\vec{k}\cdot\vec{r})^4 \cdot k^2 + \frac{10}{33}(\vec{k}\cdot\vec{r})^2 k^4 r^2 - \frac{5}{231} k^6 r^4 \right] \cdot {}^3(\vec{r})_0 \right\}$$

$$\psi_m = \varkappa^6 j_6 \cdot \left\{ (\vec{k}\cdot\vec{r})^6 - \frac{15}{11}(\vec{k}\cdot\vec{r})^4 k^2 r^2 \right.$$
$$\left. + \frac{5}{11}(\vec{k}\cdot\vec{r})^2 k^4 r^4 - \frac{5}{231} k^6 r^6 \right\} \cdot {}^1(1)_0$$

$$\psi_n = i\varkappa^7 j_7 \cdot \left\{ \left[(\vec{k}\cdot\vec{r})^6 - \frac{15}{13}(\vec{k}\cdot\vec{r})^4 k^2 r^2 \right. \right.$$
$$\left. + \frac{45}{143}(\vec{k}\cdot\vec{r})^2 k^4 r^4 - \frac{5}{143} k^6 r^6 \right] \cdot {}^3(\vec{r})_0$$
$$+ \left[-\frac{6}{13}(\vec{k}\cdot\vec{r})^5 r^2 \right.$$
$$\left. \left. + \frac{60}{143}(\vec{k}\cdot\vec{r})^3 k^2 r^4 - \frac{10}{143}(\vec{k}\cdot\vec{r}) k^4 r^6 \right] \cdot {}^3(\vec{k})_0 \right\}$$

$$\psi_o = i\varkappa^7 j_7 \left\{ (\vec{k}\cdot\vec{r})^6 - \frac{15}{13}(\vec{k}\cdot\vec{r})^4 k^2 r^2 \right.$$
$$\left. + \frac{45}{143}(\vec{k}\cdot\vec{r})^2 k^4 r^4 - \frac{5}{143} k^6 r^6 \right\} \cdot {}^3(i\vec{k}\times\vec{r})_0$$

The operator equations which these functions satisfy, and through which they were generated, are:

$$
\begin{aligned}
H\psi_a &= 2\psi_b \\
H\psi_b &= 2\psi_a + \psi_c \\
H\psi_c &= \tfrac{2}{3}k^2\psi_b + \psi_d \\
H\psi_d &= \tfrac{1}{3}k^2\psi_c + 2\psi_e \\
H\psi_e &= \tfrac{4}{5}\psi_d + 2\psi_f \\
H\psi_f &= \tfrac{6}{5}\psi_e + \psi_g \\
H\psi_g &= \tfrac{4}{7}k^2\psi_f + \psi_h \\
H\psi_h &= \tfrac{3}{7}k^2\psi_g + 2\psi_i \\
H\psi_i &= \tfrac{8}{9}\psi_h + 2\psi_j \\
H\psi_j &= \tfrac{10}{9}\psi_i + \psi_k \\
H\psi_k &= \tfrac{6}{11}k^2\psi_j + \psi_l \\
H\psi_l &= \tfrac{5}{11}k^2\psi_k + 2\psi_m \\
H\psi_m &= \tfrac{12}{13}\psi_l + 2\psi_n \\
H\psi_n &= \tfrac{14}{13}\psi_m + \psi_o
\end{aligned}
\qquad (65a - 65n)
$$

Into the wave equation, which can be written as

$$(H-\omega)\,\psi(\vec{r}) = 0 \qquad (66)$$

the expanded $\psi(\vec{r})$ in (63) can now be substituted. When the indicated operations have been carried out, as in (65), the resulting terms can be collected in the following way:

$$\begin{aligned} 0 = & \left[-\omega\cdot a + 2\cdot b\right]\cdot\psi_a \\ & + \left[2\cdot a - \omega\cdot b + \tfrac{2}{3}k^2\cdot c\right]\cdot\psi_b \\ & + \left[\, b - \omega\cdot c + \tfrac{1}{3}k^2\cdot d\right]\cdot\psi_c \\ & + \left[\, c - \omega\cdot d + \tfrac{4}{5}\cdot e\right]\cdot\psi_d \\ & + \cdots \end{aligned} \qquad (67)$$

The functions ψ_a, ψ_b, ... , as given in (64), are all linearly independent, are in fact orthogonal in a fashion which will be described later, so that (67) can be satisfied only if each of the bracketed expressions vanishes separately. What results is

a set of simultaneous equations, linear and homogeneous in the unknown coefficients a, b, c, ... , and such a system of equations has a solution only if the corresponding secular determinant vanishes. However, in this case the secular determinant has an infinite number of rows and columns; the first ten-by-ten portion is shown here:

$$
\begin{array}{c|cccccccccc}
 & a & b & c & d & e & f & g & h & i & j \\
\hline
a & -\omega & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b & 2 & -\omega & \frac{2}{3}k^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c & \cdot & 1 & -\omega & \frac{1}{3}k^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
d & \cdot & \cdot & 1 & -\omega & \frac{4}{5} & \cdot & \cdot & \cdot & \cdot & \cdot \\
e & \cdot & \cdot & \cdot & 2 & -\omega & \frac{6}{5} & \cdot & \cdot & \cdot & \cdot \\
f & \cdot & \cdot & \cdot & \cdot & 2 & -\omega & \frac{4}{7}k^2 & \cdot & \cdot & \cdot \\
g & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -\omega & \frac{3}{7}k^2 & \cdot & \cdot \\
h & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -\omega & \frac{8}{9} & \cdot \\
i & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & -\omega & \frac{10}{9} \\
j & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & -\omega
\end{array}
\qquad (68)
$$

ORTHOGONALITY AND NORMALIZATION

It was mentioned, in connection with Equation (67), that the functions ψ_a, ψ_b, ... , given in (64), are orthogonal in a certain fashion. The hermitian scalar product of two σ-spin functions can be formed in the usual way, by multiplying corresponding components together, after first taking the complex conjugate of the components of the first function. For example, the hermitian scalar product of (27) and (53) is:

$$\begin{aligned}\left({}^3(\vec{r})_0, {}^3(\vec{k})_0\right) = \frac{1}{2}\Big\{&\left[-(x-iy)^*\right]\cdot\left[-(k_x-ik_y)\right] \\ &+ z\cdot k_z + z\, k_z \\ &+ \left[(x+iy)^*\right]\cdot\left[(k_x+ik_y)\right]\Big\} \\ = {}& \vec{k}\cdot\vec{r}\end{aligned} \qquad (69)$$

The complete list of scalar products among the four σ-spin functions is given in (70):

$$
\begin{aligned}
({}^1(1)_0, {}^1(1)_0) &= 1 \qquad (70_a - 70_j)\\
({}^3(\vec{r})_0, {}^3(\vec{r})_0) &= r^2\\
({}^3(\vec{k})_0, {}^3(\vec{k})_0) &= k^2\\
({}^3(i\vec{k}\times\vec{r})_0, {}^3(i\vec{k}\times\vec{r})_0) &= k^2r^2 - (\vec{k}\cdot\vec{r})^2\\
({}^1(1)_0, {}^3(\vec{r})_0) &= 0\\
({}^1(1)_0, {}^3(\vec{k})_0) &= 0\\
({}^1(1)_0, {}^3(i\vec{k}\times\vec{r})_0) &= 0\\
({}^3(\vec{r})_0, {}^3(\vec{k})_0) &= \vec{k}\cdot\vec{r}\\
({}^3(\vec{r})_0, {}^3(i\vec{k}\times\vec{r})_0) &= 0\\
({}^3(\vec{k})_0, {}^3(i\vec{k}\times\vec{r})_0) &= 0
\end{aligned}
$$

A limited scalar product of two functions will be defined, a hermitian scalar product in which there is summation over spin components, as in (69), and averaging or integration over the direction of the vector $\vec{r}$. In practice, the direction of $\vec{r}$ enters into (70) or the scalar factors in (64) only through the combination $\vec{k}\cdot\vec{r}$.

An angle η, between the two vectors $\vec{k}$ and $\vec{r}$, can be defined:

$$\vec{k}\cdot\vec{r} = k\,r\,\cos\eta \tag{71a}$$

$$\sin^2\eta = \frac{1}{k^2r^2}\left[k^2r^2 - (\vec{k}\cdot\vec{r})^2\right] \tag{71b}$$

Averaging over the direction of $\vec{r}$ is, in practice, equivalent to averaging over the angle η, with the appropriate weighting factor $\frac{1}{2}\sin\eta$. Examples:

$$\begin{aligned}(\psi_a, \psi_a) &= \int_{\eta=0}^{\eta=\pi} \tfrac{1}{2}\sin\eta\cdot d\eta\cdot\left\{f_0^2\right\} \\ &= f_0^2\end{aligned} \tag{72}$$

$$\begin{aligned}(\psi_{1c}, \psi_{1c}) &= \int_{\eta=0}^{\eta=\pi} \tfrac{1}{2}\sin\eta\cdot d\eta\cdot\left\{\kappa^2 f_1^2\cdot k^2r^2(1-\cos^2\eta)\right\} \\ &= \tfrac{2}{3}k^2\cdot\kappa^2 r^2 f_1^2\end{aligned} \tag{73}$$

In the sense of this limited scalar product, all the functions in (64) are orthogonal.

It is also possible to introduce a relative or limited normalization, by defining new functions, ψ_ν', related to the old functions, ψ_ν, by normalization factors N_ν:

$$\psi_\nu' \equiv N_\nu \psi_\nu \tag{74a}$$

$$(\psi_\nu', \psi_\nu') = (\kappa r)^{2n} \cdot j_n^2(\kappa r) \tag{74b}$$

$$\begin{aligned}
N_a &= 1 &&= 1 \qquad (75a - 75l)\\
N_b &= 1 &&= 1\\
N_c &= \frac{1}{k}\left(\frac{3}{2}\right)^{1/2} &&= \frac{1}{k}\cdot\left(\frac{3}{2}\right)^{1/2}\\
N_d &= \frac{1}{k^2}\cdot 3\left(\frac{1}{2}\right)^{1/2} &&= \frac{1}{k^2}\cdot\left(\frac{3^2}{2}\right)^{1/2}\\
N_e &= \frac{1}{k^2}\cdot\frac{3}{2}(5)^{1/2} &&= \frac{1}{k^2}\cdot\left(\frac{3^2\cdot 5}{2^2}\right)^{1/2}\\
N_f &= \frac{1}{k^2}\cdot\frac{5}{2}(3)^{1/2} &&= \frac{1}{k^2}\cdot\left(\frac{3\cdot 5^2}{2^2}\right)^{1/2}\\
N_g &= \frac{1}{k^3}\cdot\frac{5}{4}(21)^{1/2} &&= \frac{1}{k^3}\cdot\left(\frac{3\cdot 5^2\cdot 7}{2^2\cdot 4}\right)^{1/2}\\
N_h &= \frac{1}{k^4}\cdot\frac{35}{4} &&= \frac{1}{k^4}\cdot\left(\frac{5^2\cdot 7^2}{2^2\cdot 4}\right)^{1/2}\\
N_i &= \frac{1}{k^4}\cdot\frac{105}{8} &&= \frac{1}{k^4}\cdot\left(\frac{5^2\cdot 7^2\cdot 9}{2^2\cdot 4^2}\right)^{1/2}\\
N_j &= \frac{1}{k^4}\cdot\frac{63}{8}(5)^{1/2} &&= \frac{1}{k^4}\cdot\left(\frac{5\cdot 7^2\cdot 9^2}{2^2\cdot 4^2}\right)^{1/2}\\
N_k &= \frac{1}{k^5}\cdot\frac{21}{8}\left(\frac{165}{2}\right)^{1/2} &&= \frac{1}{k^5}\cdot\left(\frac{5\cdot 7^2\cdot 9^2\cdot 11}{2^2\cdot 4^2\cdot 6}\right)^{1/2}\\
N_l &= \frac{1}{k^6}\cdot\frac{231}{8}\left(\frac{3}{2}\right)^{1/2} &&= \frac{1}{k^6}\cdot\left(\frac{7^2\cdot 9^2\cdot 11^2}{2^2\cdot 4^2\cdot 6}\right)^{1/2}
\end{aligned}$$

It will be noted that the normalization factors follow a definite pattern of modulus four. The functions (64) also follow a similar pattern, with the four σ-spin functions reappearing cyclically. The first, fifth, ninth, and thirteenth functions have the angular dependence upon $\cos\eta$ of the Legendre polynomials P_0, P_2, P_4, and P_6, respectively, and the corresponding normalization factors in (75) contain numerical factors which are just those needed to normalize the Legendre polynomials. The polynomials in the other functions of (64) can be considered to be generalizations of the Legendre polynomials.

An attempt to replace the relative normalization in (72-75) by an absolute normalization would encounter a special difficulty. The asymptotic behavior of the spherical bessel functions is:

$$j_{2m}(\varkappa r) \sim (-1)^m \cdot \frac{\sin \varkappa r}{(\varkappa r)^{2m+1}} \qquad (76a)$$

$$j_{2m+1}(\varkappa r) \sim (-1)^{m+1} \cdot \frac{\cos \varkappa r}{(\varkappa r)^{2m+2}} \qquad (76b)$$

Thus a radial integration of (74b) would diverge:

$$\int_0^{R_0 \to \infty} r^2 \cdot dr \cdot (\psi'_\nu, \psi'_\nu) \sim \frac{1}{\kappa^2} \cdot R_0 \to \infty \qquad (77)$$

In Appendix D it is shown that even the inclusion of a very small mass in the primitive wave equation will <u>not</u> produce convergence of integrations like (77).

The asymptotic behavior in (76), in fact the general form of the functions (64), can serve as evidence that the primitive waves used in building the structure are free waves in space, that the structure is thus a sort of standing-wave system.

The wave function $\psi(\vec{r})$ can be represented as an expansion in terms of the new, normalized functions:

$$\psi(\vec{r}) = a' \cdot \psi'_a + b' \cdot \psi'_b + \cdots \qquad (78)$$

The new coefficients, a', b', ... , are related to the old:

$$\left.\begin{aligned} a' &= a \cdot N_a^{-1} \\ b' &= b \cdot N_b^{-1} \\ c' &= c \cdot N_c^{-1}, \text{etc.} \end{aligned}\right\} \qquad (79)$$

In place of (68) a new secular determinant is obtained, with matrix elements symmetrical about the diagonal:

$$
\begin{array}{c||cccccccccc}
 & a' & b' & c' & d' & e' & f' & g' & h' & i' & j' \\
\hline\hline
a' & -\omega & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b' & 2 & -\omega & k\left(\frac{2}{3}\right)^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
c' & \cdot & k\left(\frac{2}{3}\right)^{1/2} & -\omega & k\left(\frac{1}{3}\right)^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
d' & \cdot & \cdot & k\left(\frac{1}{3}\right)^{1/2} & -\omega & 2\left(\frac{2}{5}\right)^{1/2} & \cdot & \cdot & \cdot & \cdot & \cdot \\
e' & \cdot & \cdot & \cdot & 2\left(\frac{2}{5}\right)^{1/2} & -\omega & 2\left(\frac{3}{5}\right)^{1/2} & \cdot & \cdot & \cdot & \cdot \\
f' & \cdot & \cdot & \cdot & \cdot & 2\left(\frac{3}{5}\right)^{1/2} & -\omega & k\left(\frac{4}{7}\right)^{1/2} & \cdot & \cdot & \cdot \\
g' & \cdot & \cdot & \cdot & \cdot & \cdot & k\left(\frac{4}{7}\right)^{1/2} & -\omega & k\left(\frac{3}{7}\right)^{1/2} & \cdot & \cdot \\
h' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k\left(\frac{3}{7}\right)^{1/2} & -\omega & 2\left(\frac{4}{9}\right)^{1/2} & \cdot \\
i' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2\left(\frac{4}{9}\right)^{1/2} & -\omega & 2\left(\frac{5}{9}\right)^{1/2} \\
j' & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2\left(\frac{5}{9}\right)^{1/2} & -\omega
\end{array}
\qquad (80)
$$

SOLUTION OF INFINITE EQUATION SYSTEM

In the case of a finite equation system, the solution of the secular equation ordinarily limits the frequency, w, to a certain number of discrete eigenvalues. In the case of the infinite equation system it appears at first inspection that the equations for the coefficients a', b', ... , can be solved for any value of w. If the first coefficient is put equal to unity, then the list can be written down:

$$a' = 1 \qquad (81_a - 81_j)$$

$$b' = \frac{w}{2}$$

$$c' = \frac{k}{2}\left(\frac{3}{2}\right)^{1/2}\cdot\left\{\frac{1}{k^2}(w^2-k^2-4) + 1\right\}$$

$$d' = \frac{w}{2}\left(\frac{1}{2}\right)^{1/2}\cdot\left\{\frac{3}{k^2}(w^2-k^2-4) + 1\right\}$$

$$e' = \frac{1}{8}(5)^{1/2}\cdot\left\{\frac{3w^2}{k^2}(w^2-k^2-4) + 4\right\}$$

$$f' = \frac{w}{16}(3)^{1/2}\cdot\left\{\frac{1}{k^2}(5w^2-8)(w^2-k^2-4) + 4\right\}$$

$$g' = \frac{k}{32}(21)^{1/2}\cdot\left\{\frac{1}{k^4}\left[5w^2(w^2-4)+4k^2\right]\cdot(w^2-k^2-4) + 4\right\}$$

$$h' = \frac{w}{32}\left\{\frac{5}{k^4}(7w^2-4k^2)(w^2-4)\cdot(w^2-k^2-4) + 12\right\}$$

$$i' = \frac{3}{8}\left\{\frac{w^2}{16k^4}\left[35w^2(w^2-4)-5k^2(7w^2-24)\right](w^2-k^2-4) + 3\right\}$$

$$j' = \frac{3w(5)^{1/2}}{16}\left\{\frac{1}{48k^4}\left[7w^2(w^2-4)(9w^2-16) - k^2(63w^4-280w^2+256)\right](w^2-k^2-4) + 1\right\}$$

It is clear that Equations (81) would be greatly simplified if the frequency, w, were made to satisfy the equation:

$$w^2 = k^2 + 4 \qquad (82)$$

However, stronger arguments are needed before (82) can be considered a true limitation on w.

As an alternative approach to the eigenvalue problem, the infinite determinant can be considered to be the limit of a sequence of finite determinants, the n-th determinant being formed from the first n rows and n columns of the infinite determinant. (Either (68) or (80) can be used.) There will then be a sequence of secular equations, each equation giving a functional relationship between w and k. The first equations in the sequence are:

$$0 = D_a = -w \qquad (83_a - 83_\ell)$$

$$0 = D_b = \begin{vmatrix} -w & 2 \\ 2 & -w \end{vmatrix} = w^2 - 4$$

$$0 = D_c = \begin{vmatrix} -w & 2 & \cdot \\ 2 & -w & k\left(\frac{2}{3}\right)^{1/2} \\ \cdot & k\left(\frac{2}{3}\right)^{1/2} & -w \end{vmatrix} = -w\left(w^2 - \frac{2}{3}k^2 - 4\right)$$

$$0 = D_d = \omega^2(\omega^2 - k^2 - 4) + \tfrac{4}{3}k^2$$

$$0 = D_e = -\omega\left\{(\omega^2 - \tfrac{8}{5})(\omega^2 - k^2 - 4) + \tfrac{4}{5}k^2\right\}$$

$$0 = D_f = (\omega^4 - 4\omega^2 + \tfrac{4}{5}k^2)\cdot(\omega^2 - k^2 - 4) + \tfrac{4}{5}k^4$$

$$0 = D_g = -\omega\left\{(\omega^4 - 4\omega^2 - \tfrac{4}{7}k^2\omega^2 + \tfrac{12}{7}k^2)(\omega^2 - k^2 - 4) + \tfrac{12}{35}k^4\right\}$$

$$0 = D_h = \omega^4(\omega^2 - k^2 - 4)^2 + \tfrac{24}{7}k^2\omega^2(\omega^2 - k^2 - 4) + \tfrac{48}{35}k^4$$

$$0 = D_i = -\omega\left\{(\omega^4 - \tfrac{16}{9}\omega^2)(\omega^2 - k^2 - 4)^2 + (\tfrac{8}{3}k^2\omega^2 - \tfrac{64}{21}k^2)(\omega^2 - k^2 - 4) + \tfrac{16}{21}k^4\right\}$$

$$0 = D_j = \omega^4(\omega^2 - 4)(\omega^2 - k^2 - 4)^2 + (\tfrac{8}{3}k^2\omega^4 - \tfrac{32}{3}k^2\omega^2 + \tfrac{16}{21}k^4)(\omega^2 - k^2 - 4) + \tfrac{16}{21}k^6$$

$$0 = D_k = -\omega\left\{(\omega^6 - 4\omega^4 - \tfrac{6}{11}k^2\omega^4 + \tfrac{32}{33}k^2\omega^2)(\omega^2 - k^2 - 4)^2 + (\tfrac{8}{3}k^2\omega^4 - \tfrac{16}{11}k^4\omega^2 - \tfrac{32}{3}k^2\omega^2 + \tfrac{80}{33}k^4)(\omega^2 - k^2 - 4) + \tfrac{80}{231}k^6\right\}$$

$$0 = D_l = \omega^6(\omega^2 - k^2 - 4)^3 + \tfrac{60}{11}k^2\omega^4(\omega^2 - k^2 - 4)^2 + \tfrac{80}{11}k^4\omega^2(\omega^2 - k^2 - 4) + \tfrac{320}{231}\cdot k^6$$

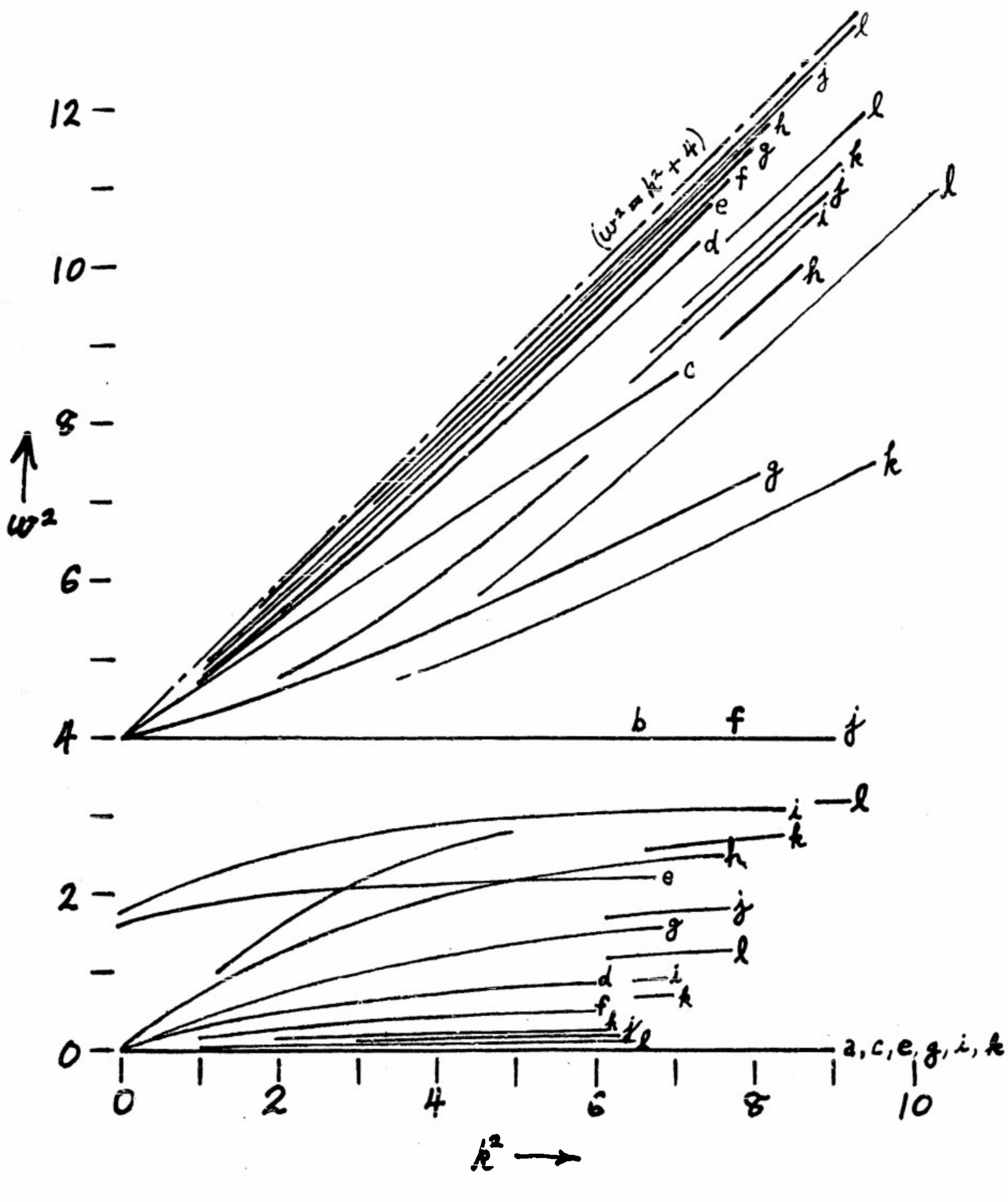

Figure 1. Solutions of secular equations (83).

The solutions of the sequence of secular equations, as plotted in Figure 1, can be seen to group themselves into several sequences of curves. Each sequence of curves appears to be approaching one or the other of the following straight lines:

$$\omega^2 = k^2 + 4 \qquad (84a)$$

$$\omega = 0 \qquad (84b)$$

Each time the size of the secular determinant is increased by four more rows and columns, new sequences of curves are started and are added to the other sequences already formed. But each new sequence sets out in much the same way as its predecessors, in the direction of (84a) or (84b).

In a geometric sense, (84a) and (84b) are the limiting solutions of the sequence of secular equations, and will therefore be considered to be the solutions of the infinite secular equation. However, it would be desirable to have an analytic proof that (84a) and (84b) are the limiting solutions of the sequence (83), and in the next section an analytic proof by induction is given.

INDUCTIVE GENERALIZATION

The close connection between some of the functions (64) and the Legendre polynomials has already been pointed out. The even-numbered Legendre polynomials are defined[4] as:

$$P_{2N}(z) = \frac{(4N)!}{2^{2N}\cdot[(2N)!]^2}\left\{ z^{2N} - \frac{(2N)(2N-1)}{2\cdot(4N-1)}\cdot z^{2N-2} + \frac{(2N)\cdot(2N-1)\cdot(2N-2)\cdot(2N-3)}{2\cdot 4\cdot(4N-1)\cdot(4N-3)}\cdot z^{2N-4} + \cdots \right\}$$

$$= \sum_{r=0}^{N} (-1)^r \frac{(4N-2r)!\cdot z^{2N-2r}}{2^{2N}\cdot r!\cdot(2N-r)!\cdot(2N-2r)!} \qquad (85)$$

The first, fifth, ninth, etc., functions may be written:

$$\left.\begin{aligned} \psi'_a &\equiv \psi'_1 = j_0(kr)\cdot P_0(\cos\eta)\cdot {}^1(1)_0 \\ \psi'_e &\equiv \psi'_5 = k^2 r^2\cdot j_2\cdot 5^{1/2}\cdot P_2(\cos\eta)\cdot {}^1(1)_0 \\ \psi'_i &\equiv \psi'_9 = k^4 r^4\cdot j_4\cdot 9^{1/2}\cdot P_4(\cos\eta)\cdot {}^1(1)_0 \end{aligned}\right\} \qquad (86a)$$

$$\psi'_{4N+1} = (kr)^{2N}\cdot j_{2N}\cdot(4N+1)^{1/2}\cdot P_{2N}(\cos\eta)\cdot {}^1(1)_0 \qquad (86b)$$

Similarly generalized expressions can be obtained for the other functions:

$$\psi'_{4N+2} = i\kappa\cdot(\kappa r)^{2N}\cdot j_{2N+1}\cdot\frac{1}{(2N+1)^{1/2}}\cdot$$
$$\cdot\left\{\left[(2N+1)\cdot P_{2N}(\cos\eta)+\cos\eta\frac{d}{d(\cos\eta)}P_{2N}(\cos\eta)\right]\cdot{}^{3}(\vec{r}_1)_0\right.$$
$$\left.+\left[-\frac{r}{k}\cdot\frac{d}{d(\cos\eta)}\cdot P_{2N}(\cos\eta)\right]\cdot{}^{3}(\vec{k})_0\right\} \qquad (87)$$

$$\psi'_{4N+3} = i\kappa(\kappa r)^{2N}\cdot j_{2N+1}\cdot\left(\frac{(4N+3)}{(2N+1)(2N+3)}\right)^{1/2}\cdot$$
$$\cdot\left\{(2N+1)P_{2N}(\cos\eta)+\cos\eta\frac{d}{d(\cos\eta)}P_{2N}(\cos\eta)\right\}\cdot\frac{1}{k}\cdot{}^{3}(i\vec{k}\times\vec{r})_0 \qquad (88)$$

$$\psi'_{4N+4} = -i(\kappa r)^{2N+1}\cdot j_{2N+1}\cdot\frac{1}{(2N+2)^{1/2}}\cdot$$
$$\cdot\left\{\left[\frac{d}{d(\cos\eta)}P_{2N+2}(\cos\eta)\right]\cdot\frac{1}{k}\cdot{}^{3}(\vec{k})_0\right.$$
$$+\left[(2N+2)P_{2N+2}(\cos\eta)\right.$$
$$\left.\left.-\cos\eta\cdot\frac{d}{d(\cos\eta)}P_{2N+2}(\cos\eta)\right]\cdot\frac{1}{r}\cdot{}^{3}(\vec{r})_0\right\} \qquad (89)$$

A generalized section of the matrix (80) can be obtained by substituting (86-89) into (63) and (66):

$$
\begin{array}{c|ccccc}
 & 4N & 4N+1 & 4N+2 & 4N+3 & 4N+4 \\
\hline
4N & -w & 2\cdot\left(\frac{2N}{4N+1}\right)^{1/2} & \cdot & \cdot & \cdot \\
4N+1 & 2\cdot\left(\frac{2N}{4N+1}\right)^{1/2} & -w & 2\cdot\left(\frac{2N+1}{4N+1}\right)^{1/2} & \cdot & \cdot \\
4N+2 & \cdot & 2\cdot\left(\frac{2N+1}{4N+1}\right)^{1/2} & -w & k\cdot\left(\frac{2N+2}{4N+3}\right)^{1/2} & \cdot \\
4N+3 & \cdot & \cdot & k\cdot\left(\frac{2N+2}{4N+3}\right)^{1/2} & -w & k\cdot\left(\frac{2N+1}{4N+3}\right)^{1/2} \\
4N+4 & \cdot & \cdot & \cdot & k\cdot\left(\frac{2N+1}{4N+3}\right)^{1/2} & -w
\end{array}
\qquad (90)
$$

The operator equations on which (90) is based are these:

$$
\left.
\begin{aligned}
H\psi'_{4N} &= k\cdot\left(\frac{2N-1}{4N-1}\right)^{1/2}\cdot\psi'_{4N-1} + 2\cdot\left(\frac{2N}{4N+1}\right)^{1/2}\cdot\psi'_{4N+1} \\
H\psi'_{4N+1} &= 2\cdot\left(\frac{2N}{4N+1}\right)^{1/2}\cdot\psi'_{4N} + 2\cdot\left(\frac{2N+1}{4N+1}\right)^{1/2}\cdot\psi'_{4N+2} \\
H\psi'_{4N+2} &= 2\cdot\left(\frac{2N+1}{4N+1}\right)^{1/2}\cdot\psi'_{4N+1} + k\cdot\left(\frac{2N+2}{4N+3}\right)^{1/2}\cdot\psi'_{4N+3} \\
H\psi'_{4N+3} &= k\cdot\left(\frac{2N+2}{4N+3}\right)^{1/2}\cdot\psi'_{4N+2} + k\cdot\left(\frac{2N+1}{4N+3}\right)^{1/2}\cdot\psi'_{4N+4} \\
H\psi'_{4N+4} &= k\cdot\left(\frac{2N+1}{4N+3}\right)^{1/2}\cdot\psi'_{4N+3} + 2\cdot\left(\frac{2N+2}{4N+5}\right)^{1/2}\cdot\psi'_{4N+5}
\end{aligned}
\right\}
\qquad (91)
$$

If the successive secular equations in the sequence (83) are obtained through the usual method for evaluating determinants[5], then a number of recurrence relations are easily derived:

$$\left.\begin{aligned}
D_{4N+1} &= -\omega\cdot D_{4N} - 4\cdot\frac{2N}{4N+1}\cdot D_{4N-1} \\
D_{4N+2} &= -\omega\cdot D_{4N+1} - 4\cdot\frac{2N+1}{4N+1}\cdot D_{4N} \\
D_{4N+3} &= -\omega\cdot D_{4N+2} - k^2\cdot\frac{2N+2}{4N+3}\cdot D_{4N+1} \\
D_{4N+4} &= -\omega\cdot D_{4N+3} - k^2\cdot\frac{2N+1}{4N+3}\cdot D_{4N+2}
\end{aligned}\right\} \qquad (92)$$

The relations (92) can be combined and manipulated to give other recurrence relations such as (93):

$$D_{4N+8} = \left[\omega^2(\omega^2-k^2-4) + 4k^2\frac{(8\cdot N^2+20\cdot N+11)}{(4N+3)\cdot(4N+7)}\right]\cdot D_{4N+4}$$

$$-16\cdot k^4\cdot\frac{(2N+1)^2\cdot(2N+2)^2}{(4N+1)(4N+3)^2(4N+5)}\cdot D_{4N} \qquad (93)$$

In the sequence (83) of secular equations, the fourth, eighth, and twelfth take a particularly simple form:

$$0 = D_{4N} = (-4k^2)^N \cdot B_N(\alpha^2) \tag{94a}$$

where:

$$\alpha^2 \equiv \frac{\omega^2 \cdot (\omega^2 - k^2 - 4)}{-4k^2} \tag{95}$$

and $B_N(\alpha^2)$ is a polynomial of degree N, in α^2. Substitution of (94a) and (95) into (93) gives a recurrence relation among the polynomials B_N:

$$B_{N+2} = \left[\alpha^2 - \frac{(8\cdot N^2 + 20\cdot N + 11)}{(4N+3)\cdot(4N+7)}\right]\cdot B_{N+1} - \frac{(2N+1)^2\cdot(2N+2)^2}{(4N+1)\cdot(4N+3)^2\cdot(4N+5)}\cdot B_N \tag{96}$$

The first few polynomials B_N are:

$$\left.\begin{aligned} B_1(\alpha^2) &= \alpha^2 - \tfrac{1}{3} \\ B_2(\alpha^2) &= \alpha^4 - \tfrac{6}{7}\alpha^2 + \tfrac{3}{35} \\ B_3(\alpha^2) &= \alpha^6 - \tfrac{15}{11}\cdot\alpha^4 + \tfrac{5}{11}\cdot\alpha^2 - \tfrac{5}{231} \end{aligned}\right\} \tag{97}$$

Apart from normalization factors, the polynomials B_N are Legendre polynomials:

$$B_N(\alpha^2) = \frac{2^{2N}\cdot[(2N)!]^2}{(4N)!}\cdot P_{2N}(\alpha) \tag{98}$$

While a completely rigorous discussion will not be attempted here, it should be clear from Figure 1 that the solutions of the secular equations can be classified into separate sequences of curves, each sequence apparently approaching a limiting curve. For example, in the upper part of Figure 1 there is a sequence of curves labeled b,c,d,e,f,g,h,(i),j,(k),ℓ. A limit for this sequence might be defined by:

$$\left.\begin{array}{ll} \omega^2 = g_\nu^{(2)}(k^2) & \nu = 2,3,4,\ldots \\ \quad \rightarrow g_\infty^{(2)}(k^2) & \\ \left| g_\nu^{(2)}(k^2) - g_{\nu+1}^{(2)}(k^2) \right| \rightarrow 0 & (\nu \rightarrow \infty) \end{array}\right\} \qquad (99a)$$

There is also a similar sequence, in the upper part of Figure 1, labeled f,g,h,i,j,k,ℓ, with a limit:

$$\left.\begin{array}{ll} \omega^2 = g_\nu^{(6)}(k^2) & \nu = 6,7,8,\ldots \\ \quad \rightarrow g_\infty^{(6)}(k^2) & \\ \left| g_\nu^{(6)}(k^2) - g_{\nu+1}^{(6)}(k^2) \right| \rightarrow 0 & (\nu \rightarrow \infty) \end{array}\right\} \qquad (99b)$$

The beginning of a third such sequence, $g_\nu^{(10)}(k^2)$, is shown by the curves labeled j,k,ℓ. In the general case, the limiting curve will be defined by:

$$\left.\begin{aligned} w^2 &\equiv g_\nu^{(4N+2)}(k^2) \qquad \nu = 4N+2,\ 4N+3,\ 4N+4,\ \ldots \\ &\rightarrow g_\infty^{(4N+2)}(k^2) \\ \left| g_\nu^{(4N+2)}(k^2) \right. &- \left. g_{\nu+1}^{(4N+2)}(k^2) \right| \rightarrow 0 \qquad (\nu \rightarrow \infty) \end{aligned}\right\} \quad (99c)$$

In the lower part of Figure 1, the curves do not fall into sequences like (99), with ν taking successive integral values. Instead the curves group themselves into even and odd sequences. For example, there is the even sequence labeled by b,d,f,h,j,ℓ:

$$w^2 = h_\nu^{(2)}(k^2) \qquad \nu = 2, 4, 6, \ldots \tag{100a}$$

There is also the odd sequence, e,g,i,k:

$$w^2 = h_\nu^{(5)}(k^2) \qquad \nu = 5, 7, 9, \ldots \tag{100b}$$

In addition, each odd secular equation includes the root w = 0:

$$w = f_\nu^{(1)}(k^2) = 0 \qquad \nu = 1, 3, 5, \ldots \tag{100c}$$

Among the upper curves of Figure 1 (the sequences in (99)), those curves which are solutions of the equations

$$D_{4N+1} = 0 \tag{94b}$$

$$D_{4N+2} = 0 \tag{94c}$$

$$D_{4N+3} = 0 \tag{94d}$$

lie bracketed between curves which are solutions of (94a). (An analytic proof of this geometrical statement will not be attempted here, but could doubtless be found from Equations (92-98).) Thus to show that the upper curves approach the limiting curve (84a) it is enough to show that corresponding roots of successive polynomials B_N approach the limit $\alpha^2 = 0$ as N approaches infinity. The identification of "corresponding" roots of (94a) can be made with the help of the roots of (94b,c,d). Figure 1, with Equation (95), shows that those roots of $B_N(\alpha^2)$ for which α^2 has its numerically smallest value are the roots which give the curves in the sequence (99a). Therefore the smallest roots of B_N are "corresponding" roots. Similarly the next smallest roots of B_N define curves which lie in the sequence (99b). In this way it is determined that the roots of two polynomials B_N and B_{N+1} are to be set in correspondence by counting upward from the smallest root of each polynomial.

Approximate values for the two smallest roots of the general polynomial B_N can be calculated from the last three terms, the earlier terms being neglected:

$$0 = B_N(\alpha^2) = \frac{(-1)^N \cdot [(2N)!]^3}{(4N)! \cdot [N!]^2} \cdot \left\{ \cdots + \right.$$

$$+ \alpha^4 \cdot \frac{(2N+3)(2N+1)(N)(N-1)}{6}$$

$$\left. - \alpha^2 \cdot N \cdot (2N+1) + 1 \right\} \qquad (101a)$$

$$\alpha^2 \doteq \frac{3}{(2N+3)(N-1)} \left\{ 1 \pm \left(\frac{1}{3} + \frac{2}{N \cdot (2N+1)} \right)^{1/2} \right\} \qquad (101b)$$

The two approximate roots in (101b) both approach zero as N approaches infinity, as was to be shown. The higher roots of (101a) can also be examined. For large values of N, Equation (101a) can be written:

$$0 \doteq 1 - 2(N^2\alpha^2) + \frac{2}{3}(N^2\alpha^2)^2 - \frac{4}{45}(N^2\alpha^2)^3$$

$$+ \frac{2}{315}(N^2\alpha^2)^4 - \cdots \qquad (102)$$

The approximation in (102) consists in omitting factors of the form:

$$\left[1 + O\left(\frac{1}{N}\right) \right] \tag{103}$$

Such factors should multiply each power of $(N^2\alpha^2)$ in (102), but each of these factors approaches unity as N becomes very large. The polynomial equation (102) will have a set of numerical roots, values of $(N^2\alpha^2)$ which cause the (infinite) polynomial to vanish. The smallest of these roots is:

$$N^2\alpha^2 \doteq 0.617 \tag{104}$$

As N becomes infinite, (102) becomes an increasingly better approximation to (101a), so that:

$$\alpha^2 \doteq \frac{\text{constant}}{N^2} \longrightarrow 0 \tag{105}$$

where the constant in (105) is a numerical root of (102) and does not depend on N.

Equations (105) and (95) show that the limit of each sequence of curves in the upper part of Figure 1 is the straight line (84a). A similar discussion can be applied to the <u>even</u> sequences in the lower part of Figure 1, because the curves satisfying (94c) are bracketed between the solutions of (94a) and can be used to determine which are "corresponding" solutions of (94a). The same rule is found for the lower curves as was found for the upper curves: the roots of $B_N(\alpha^2)$ are to be counted upward from the smallest value of α^2. The analysis in Equations (101-105) can be used again, and (95) shows that (105), as applied to the lower part of Figure 1, is equivalent to (84b). That is, the even sequences like (100a) approach the limiting straight line $w = 0$. Yet a true solution of the infinite secular determinant should be characterized by the condition

$$\left| \omega_{\nu}(k^2) - \omega_{\nu+1}(k^2) \right| \to 0 \qquad \left(\nu \to \infty \ ; \ \omega_{\nu}(k^2) \to \omega_{\infty}(k^2)\right) \qquad (105')$$

so that both odd and even equations must be included. Fortunately, the limiting solution of the even sequences is already a solution of each odd equation, as shown in (100c), so that (84b), like (84a), is a solution satisfying the requirement (105').

The eigenfunctions are found from (84) and (81):

$(\omega^2 = k^2 + 4)$:

$$
\begin{aligned}
a' &= 1 \\
b' &= \frac{\omega}{2} \\
c' &= \frac{k}{2}\cdot\left(\frac{3}{2}\right)^{1/2} \\
d' &= \frac{\omega}{2}\cdot\left(\frac{1}{2}\right)^{1/2} \\
e' &= \left(\frac{5}{2^2}\right)^{1/2} \\
f' &= \frac{\omega}{2}\cdot\left(\frac{3}{2^2}\right)^{1/2} \\
g' &= \frac{k}{2}\cdot\left(\frac{3\cdot 7}{2^2\cdot 4}\right)^{1/2} \\
h' &= \frac{\omega}{2}\cdot\left(\frac{3^2}{2^2\cdot 4}\right)^{1/2} \\
i' &= \left(\frac{3^2\cdot 9}{2^2\cdot 4^2}\right)^{1/2} \\
j' &= \frac{\omega}{2}\cdot\left(\frac{3^2\cdot 5}{2^2\cdot 4^2}\right)^{1/2} \\
k' &= \frac{k}{2}\cdot\left(\frac{3^2\cdot 5\cdot 11}{2^2\cdot 4^2\cdot 6}\right)^{1/2} \\
l' &= \frac{\omega}{2}\cdot\left(\frac{3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6}\right)^{1/2} \\
m' &= \left(\frac{3^2\cdot 5^2\cdot 13}{2^2\cdot 4^2\cdot 6^2}\right)^{1/2} \\
n' &= \frac{\omega}{2}\cdot\left(\frac{3^2\cdot 5^2\cdot 7}{2^2\cdot 4^2\cdot 6^2}\right)^{1/2}
\end{aligned}
\qquad (106a - 106n)
$$

$(w = 0)$:

$$a' = 1 \qquad (107a - 107n)$$

$$b' = 0$$

$$c' = -\frac{2}{k}\cdot\left(\frac{3}{2}\right)^{1/2}$$

$$d' = 0$$

$$e' = \left(\frac{5}{2^2}\right)^{1/2}$$

$$f' = 0$$

$$g' = -\frac{2}{k}\cdot\left(\frac{3\cdot 7}{2^2\cdot 4}\right)^{1/2}$$

$$h' = 0$$

$$i' = \left(\frac{3^2\cdot 9}{2^2\cdot 4^2}\right)^{1/2}$$

$$j' = 0$$

$$k' = -\frac{2}{k}\cdot\left(\frac{3^2\cdot 5\cdot 11}{2^2\cdot 4^2\cdot 6}\right)^{1/2}$$

$$l' = 0$$

$$m' = \left(\frac{3^2\cdot 5^2\cdot 13}{2^2\cdot 4^2\cdot 6^2}\right)^{1/2}$$

$$n' = 0$$

The coefficients in (106) and (107) can also be written in generalized notation:

$(w^2 = k^2 + 4)$:

$$\left.\begin{aligned}
(4N+1)' &= \left(\frac{1^2\cdot 3^2\cdot 5^2\cdots(2N-1)^2\cdot(4N+1)}{2^2\cdot 4^2\cdot 6^2\cdots\ (2N)^2}\right)^{1/2}\\
(4N+2)' &= \frac{w}{2}\left(\frac{1^2\cdot 3^2\cdot 5^2\cdots(2N-1)^2\cdot(2N+1)}{2^2\cdot 4^2\cdot 6^2\cdots\ (2N)^2}\right)^{1/2}\\
(4N+3)' &= \frac{k}{2}\left(\frac{1^2\cdot 3^2\cdot 5^2\cdots(2N-1)^2\cdot(2N+1)\cdot(4N+3)}{2^2\cdot 4^2\cdot 6^2\cdots\ (2N)^2\cdot(2N+2)}\right)^{1/2}\\
(4N+4)' &= \frac{w}{2}\left(\frac{1^2\cdot 3^2\cdot 5^2\cdots(2N-1)^2\cdot(2N+1)^2}{2^2\cdot 4^2\cdot 6^2\cdots\ (2N)^2\cdot(2N+2)}\right)^{1/2}
\end{aligned}\right\}\qquad(108)$$

$(w = 0)$:

$$\left.\begin{aligned}
(4N+1)' &= \left(\frac{1^2\cdot 3^2\cdot 5^2\cdots(2N-1)^2\cdot(4N+1)}{2^2\cdot 4^2\cdot 6^2\cdots\ (2N)^2}\right)\\
(4N+2)' &= 0\\
(4N+3)' &= -\frac{2}{k}\left(\frac{1^2\cdot 3^2\cdot 5^2\cdots(2N-1)^2\cdot(2N+1)\cdot(4N+3)}{2^2\cdot 4^2\cdot 6^2\cdots\ (2N)^2\cdot(2N+2)}\right)^{1/2}\\
(4N+4)' &= 0
\end{aligned}\right\}\qquad(109)$$

In (108) and (109), on the left-hand sides, the coefficients a', b', ... , have been translated to the numerical notation (1)', (2)', ... , which can more readily be written in generalized form. As N is allowed to approach infinity, the coefficients approach the following limiting values[6]:

$$\left.\begin{aligned}
&(w^2 = k^2 + 4): \qquad (N \to \infty) \\
&(4N+1)' \to 2 \cdot \left(\frac{1}{\pi}\right)^{1/2} \\
&(4N+2)' \to w \cdot \left(\frac{1}{2\pi}\right)^{1/2} \\
&(4N+3)' \to k \cdot \left(\frac{1}{\pi}\right)^{1/2} \\
&(4N+4)' \to w \cdot \left(\frac{1}{2\pi}\right)^{1/2}
\end{aligned}\right\} \qquad (110)$$

$$\left.\begin{aligned}
&(w = 0): \qquad (N \to \infty) \\
&(4N+1)' \to 2 \cdot \left(\frac{1}{\pi}\right)^{1/2} \\
&(4N+2)' = 0 \\
&(4N+3)' \to -\frac{4}{k} \left(\frac{1}{\pi}\right)^{1/2} \\
&(4N+4)' = 0
\end{aligned}\right\} \qquad (111)$$

SOLUTION IN CLOSED FORM

The solutions (84) of the infinite secular determinant were obtained from the requirement that the sequence of secular equations, formed from finite determinants of increasing size, should converge to a limiting equation. The convergence was expressed geometrically in Figure 1 and analytically in Equations (95) and (105). When the limiting solutions (84a) and (84b) are substituted into the expressions for the coefficients, the resulting values (106-111) are bounded and form converging sequences.

The coefficients and functions can be combined, as in (78), to form the eigenfunction $\psi(\vec{r})$ belonging to either of the two eigenvalues (84a) and (84b). If the spherical bessel functions $j_n(\kappa r)$ are expanded in power series, then a regrouping of terms allows the complete eigenfunction to be expressed in closed form, in terms of the ordinary bessel functions:

$(\omega^2 = k^2 + 4)$:

$$\psi(\vec{r}) = \Big\{ J_0(\varkappa r \sin\eta) \cdot {}^1(1)_0$$

$$+ \tfrac{1}{2} i\varkappa \frac{J_1(\varkappa r \sin\eta)}{(\varkappa r \sin\eta)} \cdot {}^3(i\vec{k}\times\vec{r})_0$$

$$+ \frac{\omega}{2} \cdot i\varkappa \cdot \frac{J_1(\varkappa r \sin\eta)}{(\varkappa r \sin\eta)} \cdot \left[{}^3(\vec{r})_0 - \frac{(\vec{k}\cdot\vec{r})}{k^2} \cdot {}^3(\vec{k})_0 \right] \Big\} \qquad (112)$$

$(\omega = 0)$:

$$\psi(\vec{r}) = \Big\{ J_0(\varkappa r \sin\eta) \cdot {}^1(1)_0$$

$$- \frac{2i\varkappa}{k^2} \cdot \frac{J_1(\varkappa r \sin\eta)}{(\varkappa r \sin\eta)} \cdot {}^3(i\vec{k}\times\vec{r})_0 \Big\} \qquad (113)$$

The angle η was defined in (71). The algebraic sign of the quantity $\sin\eta$ is not of significance in (112) and (113), since only even powers of $\sin\eta$ are involved:

$$J_0(\varkappa r \sin\eta) = 1 - \frac{\varkappa^2 r^2 \sin^2\eta}{2\cdot 2} + \frac{\varkappa^4 r^4 \sin^4\eta}{2\cdot 2\cdot 4\cdot 4} - \cdots \qquad (114)$$

$$\frac{J_1(\varkappa r \sin\eta)}{(\varkappa r \sin\eta)} = \frac{1}{2} - \frac{\varkappa^2 r^2 \sin^2\eta}{2\cdot 2\cdot 4} + \frac{\varkappa^4 r^4 \sin^4\eta}{2\cdot 2\cdot 4\cdot 4\cdot 6} - \cdots \qquad (115)$$

It can be seen that the wave functions (112) and (113) have a limited dependence on $\vec{r}$, depending only on those components of $\vec{r}$ which are perpendicular to $\vec{k}$. That is, for the plane wave solution (20) of the general wave equation, the internal-structure function $\psi(\vec{r})$ is wholly transverse in character, at least in this special case of zero spin with $\tau_1 = \tau_2 = +1$.

The first of the two solutions, (112), is of particular interest, since it describes the motion of a particle with mass. Through (51), Equation (84a) can be written:

$$K_0^2 = K^2 + (2\varkappa)^2 \qquad (116)$$

Equation (116) is the desired generalization of (32), and reduces to (32) when the center of gravity is at rest. Equation (116) is the familiar Einstein relationship between energy, momentum, and mass, translated to the wave terms of frequency, propagation wavenumber, and rest frequency, all expressed in wavenumber units.

The second solution, (113), is a static solution, as shown by its corresponding secular equation (84b). This zero-frequency solution does not show particle behavior.

As written in (113), the second solution does not remain finite as k is allowed to approach zero. However, the wave function (113) can easily be renormalized to keep it finite, simply by multiplying it by k or k^2.

Discussion of the physical significance of the zero-frequency solution (113) will be deferred, but the particle behavior of (112) gives rise to the question of its identification among the known elementary particles. The fact that (112) is a zero-spin structure of pseudoscalar character (an inversion of coordinates will interchange the two waves of the double-wave system, reversing the sign of the internal-structure wave function) suggests that (112) might represent a pi-meson. It might be supposed that the charge of the particle could be related to the τ-dependence of the structure, perhaps with charged structures having $\tau_1 = \tau_2$, while neutral structures had $\tau_1 = -\tau_2$. However, such a speculation would be confronted by the results of Appendix C, in which it is shown that exactly the same roots, (84a) and (84b), are obtained, no matter which τ-dependence is used. Thus (112) cannot be identified as a pi-meson, because there would then be no way of accounting for the observed mass difference between the charged pi-meson and the neutral pi-meson.

Nevertheless, from the study of other structures, to be discussed in separate chapters, there is reason to believe that the structure (112) is closely related to the pi-meson, the charged pi-meson probably being a four-wave structure which combines the two-wave structure (112) with a different two-wave structure (the photon), to be described in another chapter. If it is tentatively assumed that (112) is a close cousin of the pi-meson, and has approximately the same mass, then a numerical value can be given to the rest-wavenumber $2\varkappa$, which represents the particle mass. The wavelength, which would then correspond to the radian wavenumber $\varkappa$, is:

$$\frac{2\pi}{\varkappa} = 2.8 \cdot 10^{-13}\ \text{cm.} \tag{117}$$

It is interesting to note that, if the magnitude of $\vec{k}$ is allowed to approach zero, then (112) remains dependent upon the direction of $\vec{k}$. If, at the same time, (112) is averaged over the direction of $\vec{k}$, then (112) reduces to the solution which was previously obtained, in (37), for the special case in which the center of gravity was at rest.

The construction of a wave packet from the plane wave solutions will be discussed in Appendix A.

METHOD OF UNDETERMINED MULTIPLIERS

The secular equations (84a) and (84b) were obtained from the infinite secular determinant (80). Both a geometric and an analytic method were used, but neither of these two methods appears suitable for generalization to the much more complicated secular determinants which will appear later in the theoretical development, when spin-one structures and three- and four-wave systems are analyzed. An alternative procedure of greater simplicity, to be called the method of undetermined multipliers, will be described here and will be used in later analyses.

It can be seen from (108) and (109) that the coefficients in the expansion of an eigenfunction fall into four sets. Within any one of the four sets, the ratio of any two of the coefficients is a pure number, with no dependence on k or w. And a comparison of (108) with (109) shows that the ratio is the same for both eigenfunctions, although this will not always happen with other structures.

In each structure that is analyzed it will be found that the functions and the coefficients can be grouped naturally into a finite number of such sets. In the present case the number of sets is four, but for other structures

the number of sets may differ. In general the number of sets is equal to the number of independent σ-spin functions.

With the method of undetermined multipliers the infinite secular determinant is reduced to a finite secular determinant of dimension equal to the number of sets of functions or coefficients. In the present case the reduction is to a four-by-four determinant. Only one coefficient from each set is used, the higher coefficients being replaced by numerical multiples of the first coefficient in the appropriate set, wherever it is necessary to use equations involving the higher coefficients. The undetermined multipliers are then fixed by requiring that the secular equation take one of the following two forms:

$$\omega^2 = k^2 + \text{constant} \tag{118a}$$

$$\omega = 0 \tag{118b}$$

In the present case, the fifth coefficient will be written as a multiple of the first coefficient:

$$e' = E \cdot a' \tag{119}$$

With the replacement (119), the first four simultaneous equations can be written with only four coefficients explicitly involved:

$$\left.\begin{aligned}
-\omega\cdot a' + 2\cdot b' &= 0\\
2\cdot a' - \omega\cdot b' + k\cdot\left(\tfrac{2}{3}\right)^{1/2}\cdot c' &= 0\\
k\cdot\left(\tfrac{2}{3}\right)^{1/2}\cdot b' - \omega\cdot c' + k\cdot\left(\tfrac{1}{3}\right)^{1/2}\cdot d' &= 0\\
2\cdot\left(\tfrac{2}{5}\right)^{1/2}\cdot E\cdot a' + k\cdot\left(\tfrac{1}{3}\right)^{1/2}\cdot c' - \omega\cdot d' &= 0
\end{aligned}\right\} \quad (120)$$

There will be a solution for a', b', c', d', in (120), only if the secular determinant vanishes:

$$0 = \omega^2(\omega^2 - k^2 - 4) + \tfrac{4}{3}k^2\left[1 - E\cdot 2\left(\tfrac{1}{5}\right)^{1/2}\right] \qquad (121)$$

There are solutions of the forms (118), provided that:

$$E = \tfrac{1}{2}\cdot(5)^{1/2} \qquad (122)$$

Comparison of (119) and (122) with (106-109) shows that the results obtained by this method of undetermined multipliers are the same as the results obtained by the geometric and analytic methods described previously.

Once the ratio E has been determined, as in (122), the secular equation (121) becomes

$$0 = \omega^2(\omega^2 - k^2 - 4) \tag{123}$$

which factors directly into (84a) and (84b). The individual coefficients can be found from Equations (120) and the later equations in the simultaneous system described by the matrix (80). Since the equations are homogeneous in the coefficients, one coefficient can be set arbitrarily. If the first coefficient, a', is chosen to be unity, then the whole list of coefficients will be identical with the list obtained previously.

The method of undetermined multipliers can be generalized. Instead of the minimum number of equations, as in (120), any greater number can be used, with the higher coefficients (beyond the number of equations) being replaced by multiples of lower coefficients in the same set. The secular determinant is then more complicated, but the requirement that the secular equation be of the form (118a) or (118b) leads once again to the same solutions. For example, five equations could be used, with the sixth coefficient replaced by a multiple of the second:

$$f' = F \cdot b' \tag{124}$$

$$0 = -\frac{\omega}{5}\left\{ (5\omega^2 - 8)(\omega^2 - k^2 - 4) + 4k^2 \cdot \left[1 - F \cdot 2 \cdot \left(\frac{1}{3}\right)^{1/2} \right] \right\} \tag{125}$$

$$F = \frac{1}{2}(3)^{1/2} \tag{126}$$

With the choice (126), which agrees with (106-109), there is in this case, besides the two desired solutions (118), an additional root:

$$\omega^2 = \frac{8}{5} \tag{127}$$

This root is not of the forms (118) and does not recur as the number of equations is increased. It is therefore considered a spurious root not characteristic of the infinite equation system.

Six equations could have been used instead of five or four:

$$g' = G \cdot c' \tag{128}$$

$$0 = -\frac{(\omega^2-4)}{5}\left\{ 5\,\omega^2(\omega^2-k^2-4) + 4k^2\left[1 - G\cdot 2\cdot\left(\frac{2}{7}\right)^{1/2}\right]\right\} \qquad (129)$$

$$G = \frac{1}{2}\left(\frac{7}{2}\right)^{1/2} \qquad (130)$$

Once again, in addition to the desired roots (84), there is an additional, spurious root:

$$\omega^2 = 4 \qquad (131)$$

If eight (or more) equations are used, there is then a choice in the way the higher coefficients are to be expressed, but the resulting solutions are the same, no matter which alternative is chosen. For example, the ninth coefficient may be expressed as a multiple of the first, or as a multiple of the fifth:

$$i' = I\cdot a' \qquad (132a)$$

$$i' = I'\cdot e' \qquad (132b)$$

These alternatives lead respectively to the secular equations:

$$0 = \frac{1}{35}\cdot\Big\{35\cdot\left[\omega^2(\omega^2-k^2-4)\right]^2 + 120\cdot k^2\cdot\left[\omega^2(\omega^2-k^2-4)\right] + 48\cdot k^4\cdot\left[1-\frac{8}{9}\cdot I\right]\Big\} \tag{133a}$$

$$0 = \frac{1}{35}\cdot\Big\{35\cdot\left[\omega^2(\omega^2-k^2-4)\right]^2 + 120\cdot k^2\cdot\left[\omega^2(\omega^2-k^2-4)\right] + 48\cdot k^4 - k^2\cdot I'\cdot\frac{16}{3}(5)^{1/2}\cdot\left[3\omega^2(\omega^2-k^2-4)+4k^2\right]\Big\} \tag{133b}$$

In either case, the same solutions as before are obtained in the same way as before:

$$I = \frac{9}{8} \tag{134a}$$

$$I' = \frac{9}{4}\left(\frac{1}{5}\right)^{1/2} \tag{134b}$$

It would, of course, be possible to generalize this procedure with the help of the general matrix elements in (90), but the essential simplicity of the method would thereby be lost. The practicality of the method of undetermined multipliers lies in the clarity and directness of its simplest form, as illustrated in (118-123).

THE FERMI SEA

It is clear from the form of the functions in (64), (112), and (113), that the double-wave structure is not a deuteron-like combination of two sub-particles bound together in space, since such a combination would involve a negative-exponential dependence upon the relative coordinate r, whereas the actual functions show a periodic dependence upon r. The structure is actually a standing-wave system, a partially localized configuration which has a center and an identity, and which, in the case of (112), moves as a particle, as shown in (116). However, the structure can be resolved into its elements, these elements being primitive waves extended through the universe. Each function in (64) is the combination of a converging and a diverging spherical wave, while the waves in (112) and (113) are cylindrical. The fact that the dependence upon r is periodic rather than exponential is an intrinsic feature of the theory that goes back to Equations (31), to the positive sign accompanying the constant terms there, and back further to (20) and the assumption of a periodic dependence on the laboratory time T, and to (17) and the deliberate omission of any explicit coupling between primitive waves.

It should also be noted that all the functions used in the double-wave structure have the same periodicity in space, a periodicity characterized by the wavenumber $\varkappa$. The possible physical significance of this quantity $\varkappa$, which serves as the unit of mass, is the main subject of the present section.

If a structure such as (118), satisfying the Einstein relationship (116), is to represent a particular kind of particle, with a particular mass $2\varkappa$, then several such structures, representing several particles of the same kind, ought all to have the same value of rest mass or rest frequency. The wavenumber $\varkappa$, the unit of rest mass, ought therefore to have the same value for all of the separate structures. In addition there are many other kinds of possible structures; a double-wave structure of spin one and a triple-wave structure of spin one-half will be described in separate chapters. In each case the rest mass will appear as a multiple of a wavenumber $\varkappa$, and it is important that the $\varkappa$ have the same value and significance in each case, so that numerical mass ratios can be calculated and compared with experiment.

It has already been pointed out, in the discussion of Equation (116), that the structure (112) is actually not a satisfactory model for any of the known elementary particles, but other structures will be found to be satisfactory models for certain of the elementary particles, and in these cases the arguments just given acquire a special validity. From these arguments the wavenumber κ emerges as a basic constant, the reciprocal of a fundamental length, though as yet without a physical meaning. It is possible, still without asking about physical meaning, to suggest a simple rule for building functions in the more complicated cases. If Equation (16) is used to express the relative vector, $\vec{r}$, in terms of the individual vectors, $\vec{r}_1$ and $\vec{r}_2$, then it is found that, at least for small values of $\vec{k}$, the structural functions are built from a pair of primitive waves, <u>each having the same propagation wavenumber κ.</u> In the generalization to a triple-wave structure, the form of the wave equation indicates that the scalar dependence upon the relative vectors, $\vec{r} = \frac{1}{2}(2\vec{r}_1 - \vec{r}_2 - \vec{r}_3)$ and $\vec{\rho} = (\vec{r}_2 - \vec{r}_3)$, should be a product of spherical bessel functions. Within this limitation, the wavenumber κ can be introduced in such a way that, in particularly symmetrical orientations of the three waves, the waves each have the same wavenumber κ.

In the triple-wave case the functions must be antisymmetrized, and permuted coordinates are useful:

$$\left.\begin{aligned} \vec{r} &= \tfrac{1}{2}(2\vec{r}_1 - \vec{r}_2 - \vec{r}_3) & \vec{\rho} &= (\vec{r}_2 - \vec{r}_3) \\ \vec{r}' &= \tfrac{1}{2}(2\vec{r}_2 - \vec{r}_3 - \vec{r}_1) & \vec{\rho}' &= (\vec{r}_3 - \vec{r}_1) \\ \vec{r}'' &= \tfrac{1}{2}(2\vec{r}_3 - \vec{r}_1 - \vec{r}_2) & \vec{\rho}'' &= (\vec{r}_1 - \vec{r}_2) \end{aligned}\right\} \quad (135)$$

The appropriate products of spherical bessel functions can be abbreviated:

$$\left.\begin{aligned} j_{nm} &\equiv j_n(\kappa r)\cdot j_m\left(\tfrac{\sqrt{3}}{2}\kappa\rho\right) \\ j'_{nm} &\equiv j_n(\kappa r')\cdot j_m\left(\tfrac{\sqrt{3}}{2}\kappa\rho'\right) \\ j''_{nm} &\equiv j_n(\kappa r'')\cdot j_m\left(\tfrac{\sqrt{3}}{2}\kappa\rho''\right) \end{aligned}\right\} \quad (136)$$

Two independent solutions of the triple-wave equation can be generated from the following two functions, in which the above abbreviations and the conventional spin notation are used. (These functions are included here to illustrate the multiple-wave generalizations of the double-wave functions. Detailed discussion of the triple-wave and quadruple-wave solutions will be reserved for later chapters.)

$$
\begin{aligned}
(\psi_2)^{+\frac{1}{2}}_{\frac{1}{2}} = {} & \left[\chi^+(\tau_1)\cdot\chi^+(\tau_2)\cdot\chi^+(\tau_3)\right]\cdot\Big\{\left[2 f_{oo} - f'_{oo} - f''_{oo}\right]\cdot \\
& \cdot\left[\chi^+(\sigma_1)\cdot\chi^-(\sigma_2)\cdot\chi^+(\sigma_3) - \chi^+(\sigma_1)\cdot\chi^+(\sigma_2)\cdot\chi^-(\sigma_3)\right] \\
& - \left[f'_{oo} - f''_{oo}\right]\cdot\left[2\cdot\chi^-(\sigma_1)\cdot\chi^+(\sigma_2)\cdot\chi^+(\sigma_3)\right. \\
& \left. - \chi^+(\sigma_1)\cdot\chi^-(\sigma_2)\cdot\chi^+(\sigma_3) - \chi^+(\sigma_1)\cdot\chi^+(\sigma_2)\cdot\chi^-(\sigma_3)\right]\Big\}
\end{aligned}
\tag{137}
$$

$$
\begin{aligned}
(\bar{\psi}_2)^{+\frac{1}{2}}_{\frac{1}{2}} = {} & \left[f_{oo} + f'_{oo} + f''_{oo}\right]\cdot\Big\{\left[2\cdot\chi^-(\tau_1)\cdot\chi^+(\tau_2)\cdot\chi^+(\tau_3)\right. \\
& \left. - \chi^+(\tau_1)\cdot\chi^-(\tau_2)\cdot\chi^+(\tau_3) - \chi^+(\tau_1)\cdot\chi^+(\tau_2)\cdot\chi^-(\tau_3)\right]\cdot \\
& \cdot\left[\chi^+(\sigma_1)\cdot\chi^-(\sigma_2)\cdot\chi^+(\sigma_3) - \chi^+(\sigma_1)\cdot\chi^+(\sigma_2)\cdot\chi^-(\sigma_3)\right] \\
& - \left[\chi^+(\tau_1)\cdot\chi^-(\tau_2)\cdot\chi^+(\tau_3) - \chi^+(\tau_1)\cdot\chi^+(\tau_2)\cdot\chi^-(\tau_3)\right]\cdot \\
& \cdot\left[2\cdot\chi^-(\sigma_1)\cdot\chi^+(\sigma_2)\cdot\chi^+(\sigma_3)\right. \\
& \left. - \chi^+(\sigma_1)\cdot\chi^-(\sigma_2)\cdot\chi^+(\sigma_3) - \chi^+(\sigma_1)\cdot\chi^+(\sigma_2)\cdot\chi^-(\sigma_3)\right]\Big\}
\end{aligned}
\tag{138}
$$

While it is not strictly necessary, in building structures, to attribute a physical meaning to the fundamental wavenumber κ, such a physical meaning is nevertheless desirable and can actually be constructed

through general arguments which are similar to those in the first two sections of this chapter, though perhaps less compelling. These arguments will be given in more detail in a separate chapter, but will be included here in the following abbreviated form:

Since all elementary particles are observed to combine in themselves a wave aspect and a particle aspect, it can be inferred that these are two concepts which are needed in the building of all particles and are therefore concepts which belong on the next conceptual level below the level of the elementary particles. Thus the primitive field itself should have both a wave aspect and a particle aspect, since the primitive field must include whatever is found on its conceptual level in order to ensure its singleness. Furthermore, since all the elementary particles (so far as is known) respond to gravitation, and in exactly the same way, the primitive field itself should be directly related to gravitation, and ought to respond to gravitation itself. In its particle aspect it should show particle response, in its wave aspect wave response, but these two responses should not be separate phenomena but different aspects of the same phenomenon.

The response of particles to gravitation can be visualized, for example, by considering the behavior of the molecules of the earth's atmosphere. These particles exhibit a density gradient; nearer the source of gravitation the particles congregate in greater numbers. The wave response can similarly be visualized. A wave responding to the same gravitational attraction would be bent toward the source, with the result that the successive wavefronts in the wave train would be crowded together on the lower side and spread apart on the upper side of the wave train. There would thus be a wavenumber gradient paralleling the particle density gradient. These two gradients can represent the same phenomenon from two points of view provided that the primitive field is of such a nature that a wavenumber gradient always accompanies a parallel density gradient in gravitational situations. It has already been pointed out that the primitive field should have Fermi-Dirac statistics, but the gravitational argument reinforces this conclusion, since a Fermi-Dirac field is needed to produce such a correlation between wavenumber and density: if a certain number of Fermi-Dirac waves or particles are enclosed in a box of a given size, then the

ground state is a state in which all possible levels of the field are occupied up to a certain wavenumber; and the greater the numerical density, the higher the wavenumber marking the top of the "Fermi sea".

It is not essential that the system be in its lowest state, but it is necessary that the system be reasonably close to equilibrium at a low "Fermi temperature", so that the wavenumber value marking the approximate surface of the Fermi sea can actually be defined. If the system were in a highly excited, non-equilibrium state, the desired correlation between wavenumber and numerical density could not be established. When the system is near equilibrium at a low Fermi temperature, then disturbances or deviations from equilibrium will tend to be localized, in wavenumber space, to the regions near the Fermi surface. In particular, elementary particles, viewed as excitations of the primitive field, will tend to be built from primitive waves near the Fermi surface, from primitive waves all having the same wavenumber $\varkappa$, where $\varkappa$ is the height of the Fermi sea. Thus the foregoing argument leads to the same rule for building elementary particles that was described earlier, while giving an added physical meaning to the fundamental wavenumber $\varkappa$.

However, $\varkappa$ emerges from this argument not as a universal constant but as a parameter which varies through space, being greater in those regions where the gravitational potential is higher. Since $\varkappa$ serves to set the scale of mass for the elementary particles, and $\varkappa c$ sets the scale of frequency, both masses and frequencies will also vary through space, though only gradually. In regions where there is a gradient of $\varkappa$, a standing-wave structure of the sort which has been analyzed in this chapter can still be matched to such a sloping boundary condition, but the structure thereby acquires an acceleration of gravitational character. Furthermore, the primitive waves used in the structure will themselves be refracted in passing through such a region of changing $\varkappa$, and the incorporation of refracted waves into an accelerated structure leads to gravitational effects which include the three crucial phenomena described by Einstein. The details of the gravitational calculations are given in a separate chapter.

It will be found helpful, in the discussion (in a separate chapter), of the origin or source of the gravitational gradients, to assume a non-vanishing

spacing of the levels in the Fermi sea. This is equivalent to the localization of a primitive wave within a dimension of cosmic size, of the order of two billion light-years. It is intended to effect this localization through the inclusion in the primitive wave equation of a very small mass-like term (probably imaginary), so small that the velocities of such structures as the photon and neutrino would not be appreciably changed from c, although there might be small but significant modifications of equations, which might, for example, affect the explanation of the galactic red-shift. In Appendix D there is an analysis of those modifications of the double-wave, spin-zero structure which would accompany the incorporation of a very small mass-like term in the primitive wave equation.

SUMMARY

In order to terminate the historical sequence which has led through atoms and nuclei to the elementary particles, it is postulated that there exists a single primitive wave field, ξ, whose properties are to be inferred from its singleness. The spin should be half-integral and is chosen to be one-half. The velocity should be no less than the velocity of a photon, and the mass is chosen to be either zero or so small that it would correspond to a Compton wavelength of cosmic dimension.

On the assumption that the mass is rigorously zero, a wave equation (1) is written down and a double-wave equation (17) is constructed. Two solutions with zero spin are found (112,113) and related solutions are given in Appendix C. The construction of a wave packet is described in Appendix A. An auxiliary equation (18) giving the dependence on the relative time variable is solved in Appendix B. In Appendix D are described the modifications which follow the inclusion of a very small mass in the primitive wave equation.

In this theory operators are represented by infinite matrices, and one such matrix (80) is studied in considerable detail, in order to justify a simple method (the method of undetermined multipliers) which will be used in later chapters in the reduction of other infinite matrices to finite form.

The solutions of the double-wave equation are standing-wave systems, combinations of converging and diverging primitive waves. The physical interpretation of the solutions, based partly on gravitational arguments, is that the universe is occupied by primitive waves which form a Fermi sea, filling momentum space (or wavenumber space) up to a certain level which is measured by the wavenumber $\varkappa$. Elementary particles are standing-wave structures formed from primitive waves near the surface of the Fermi sea.

If the spin zero structure (112), which moves like a particle with mass $2\varkappa$ (in wavenumber units), is tentatively identified as a near cousin to a pi-meson, then the wavelength of the primitive waves at the surface of the Fermi sea is $2\pi/\varkappa = 2.8 \cdot 10^{-13}$ cm. In addition to the particle solution (112) there is a static solution given in Equation (113).

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